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Solution to the Rarita–Schwinger equations

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Abstract. A systematical method is presented for solving the relativistic wave equations for particles of arbitrary spin. Explicit helicity relativistic wave functions for particles with arbitrary spin are derived rigorously.

1 Introduction

In the area of high energy physics, in order to calculate the relevant Feynman diagrams and perform amplitude analyses for high energy processes it is necessary to employ higher spin relativistic wave functions [1–4].

Various forms of wave equations for particles with spins greater than unity have been discussed extensively [5–9]. In view of solutions, the most convenient forms are the Klein–Gordon (K–G) equations for integral spin and the Rarita–Schwinger [8] (R–S) equations for half-integral spin. Moldauer and Case [10] once pointed out that the R–S equations can be derived from Dirac–Fierz–Pauli theory [5–7]. In the first part of the present work, we will show that both of the K–G and R–S equations can be derived from the Bargmann–Wigner [9] (B–W) equations. A profound property of this derivation is that it demonstrates clearly that all the subsidiary conditions added to the K–G and R–S equations are included in the B–W equation, and thus "the B–W equation could be regarded as the simplest, and the least restrictive (though in many ways the most profound) set of equations" [11].

The easiest method to construct explicit wave functions that satisfy the K–G equations for integral spin and R–S equations for half-integral spin was proposed by Auvil and Brehm [3, 12]. These wave functions are constructed from the spin-1/2 and spin-1 functions with Clebsch–Gordan coefficients. By always coupling to the maximum possible spin, Auvil and Brehm found that this kind of wave functions automatically satisfies the K–G or R–S equations. Based on this method, a more closed form of the wave functions for an arbitrary integral spin has been derived recently by Chung [1]. As a complete solution to the K–G or R–S equations, however, these wave functions should be derived rigorously from the K–G or R–S equations. The principal purpose of this paper is to develop a systematical method to solve the K–G and then the R–S equations; we would not only deduce rigorously explicit helicity wave functions, corresponding to positive and negative energy solutions both in momentum and in coordinate representations for arbitrary integral and half-integral spins in a step-by-step way, but also show clearly how the non-maximum spin components are removed by the subsidiary conditions contained in the K–G and R–S equations. For an arbitrary integral spin, the wave functions in the momentum representation derived from the K–G equations in this work are consistent with those given by Chung [1], except that our results are not presented in the rest frame. The procedure and the results expressed in the solution to the K–G equations form an important foundation for solving the R–S equations.

2 Wave equations for particles with higher spins

We discuss first that both of the K–G and R–S equations can be derived from the B–W equations.

2.1 The B–W and K–G equations for integral spins

2.1.1 Spin 1

The B–W equations for spin 1 are

$$(\partial\!\!\!/ + m)_{\alpha\alpha'}\Psi_{\alpha'\beta}(x) = 0, \tag{1a}$$
$$(\partial\!\!\!/ + m)_{\beta\beta'}\Psi_{\alpha\beta'}(x) = 0, \tag{1b}$$

where $\Psi_{\alpha\beta}(x)$ is a symmetric spinor of rank 2 which can be expressed as [13]

$$\Psi_{\alpha\beta}(x) = (\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu}C\partial_\mu)_{\alpha\beta}\, A^\nu(x), \tag{2}$$

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with $C = \gamma_2\gamma_4$ the charge conjugation matrix, $\gamma_\mu C$ and $\Sigma_{\mu\nu}C$ the symmetric matrices, and $A^\nu(x)$ the vector fields satisfying the well-known K–G equations

$$(\Box - m^2)A^\nu(x) = 0 \quad (\nu = 1,2,3,4), \tag{3a}$$
$$\partial_\nu A^\nu(x) = 0. \tag{3b}$$

2.1.2 Spin 2

The B–W equation for spin 2 reads

$$(\not\partial + m)_{\alpha\alpha'}\Psi_{\alpha'\beta\delta\tau}(x) = 0, \tag{4a}$$
$$(\not\partial + m)_{\beta\beta'}\Psi_{\alpha\beta'\delta\tau}(x) = 0, \tag{4b}$$
$$(\not\partial + m)_{\delta\delta'}\Psi_{\alpha\beta\delta'\tau}(x) = 0, \tag{4c}$$
$$(\not\partial + m)_{\tau\tau'}\Psi_{\alpha\beta\delta\tau'}(x) = 0, \tag{4d}$$

where $\Psi_{\alpha\beta\delta\tau}(x)$ is a completely symmetric multispinor of rank 4. Comparing (4a) and (4b) with (1a) and (1b), it is easy to see that they satisfy the same *Dirac* equation with respect to the indexes α and β; furthermore, $\Psi_{\alpha\beta\delta\tau}(x)$ is symmetric between α and β. Therefore, by using exactly the same procedure in which (2) and (3a) and (3b) are derived, we obtain

$$\Psi_{\alpha\beta\delta\tau}(x) = (\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu}C\partial_\mu)_{\alpha\beta}\, A^\nu_{\delta\tau}(x), \tag{5}$$

with $A^\nu_{\delta\tau}(x)$ satisfying the following equations:

$$(\Box - m^2)A^\nu_{\delta\tau}(x) = 0 \quad (\nu = 1,2,3,4), \tag{6a}$$
$$\partial_\nu A^\nu_{\delta\tau}(x) = 0. \tag{6b}$$

Since $\Psi_{\alpha\beta\delta\tau}(x)$ is also satisfying the *Dirac* equation (4c), we have

$$(\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu}C\partial_\mu)_{\alpha\beta}\,(\not\partial + m)_{\delta\delta'}A^\nu_{\delta'\tau}(x) = 0;$$

because α and β are arbitrary indices, this equation can be written in matrix form, namely

$$\mathrm{i}m\,(\gamma_\nu C)\,(\not\partial + m)_{\delta\delta'}A^\nu_{\delta'\tau}(x) + (\Sigma_{\mu\nu}C)\partial_\mu(\not\partial + m)_{\delta\delta'}A^\nu_{\delta'\tau}(x) = 0;$$

by virtue of the independence of the matrices $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$, the above equation gives

$$(\not\partial + m)_{\delta\delta'}A^\nu_{\delta'\tau}(x) = 0. \tag{7a}$$

Similarly, since $\Psi_{\alpha\beta\delta\tau}(x)$ is also satisfying the *Dirac* equation (4d), we have

$$(\not\partial + m)_{\tau\tau'}A^\nu_{\delta\tau'}(x) = 0. \tag{7b}$$

Equations (7a) and (7b) are similar to (1a) and (1b) with respect to the spinor indexes δ and τ; therefore, we have [refer to (2) and (3a) and (3b)]

$$A^\nu_{\delta\tau}(x) = (\mathrm{i}m\gamma_{\nu'} C + \Sigma_{\mu'\nu'}C\partial_{\mu'})_{\delta\tau}\, A^{\nu\nu'}(x), \tag{8}$$

with $A^{\nu\nu'}(x)$ satisfying the equations below:

$$(\Box - m^2)A^{\nu\nu'}(x) = 0 \quad (\nu,\nu' = 1,2,3,4), \tag{9a}$$
$$\partial_{\nu'}A^{\nu\nu'}(x) = 0. \tag{9b}$$

Substituting (8) into (6b) and applying the independence of the matrices $\gamma_{\nu'}C$ and $\Sigma_{\mu'\nu'}C$, we obtain

$$\partial_\nu A^{\nu\nu'}(x) = 0. \tag{9c}$$

If (8) is substituted into (6a), one obtains the same one as (9a). Combining (5) and (8), $\Psi_{\alpha\beta\delta\tau}(x)$ can now be expressed as

$$\Psi_{\alpha\beta\delta\tau}(x) = (\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu}C\partial_\mu)_{\alpha\beta} \times (\mathrm{i}m\gamma_{\nu'} C + \Sigma_{\mu'\nu'}C\partial_{\mu'})_{\delta\tau}\, A^{\nu\nu'}(x). \tag{10}$$

The right hand side of this expression is symmetric both in α and β, and in δ and τ. In order to make sure that this expression is completely symmetric in all the four indexes α, β, δ and τ, we require that the contraction of $\Psi_{\alpha\beta\delta\tau}(x)$ with the three independent antisymmetric *Dirac* matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ with respect to the indices β and δ vanish, namely

$$\Psi_{\alpha\beta\delta\tau}(x)(C^{-1})_{\beta\delta} = 0, \tag{11a}$$
$$\Psi_{\alpha\beta\delta\tau}(x)(C^{-1}\gamma_5)_{\beta\delta} = 0, \tag{11b}$$
$$\Psi_{\alpha\beta\delta\tau}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\beta\delta} = 0. \tag{11c}$$

These, as shown below, will yield some new conditions. Expanding (10) and substituting it into (11a) and (11b), using the formulas for the products of γ matrices and using (9b) and (9c), (11a) and (11b) become

$$(\Box - m^2)A^{\nu\nu}(x) + \mathrm{i}(-m^2)A^{\nu\nu'}(x)\Sigma_{\nu\nu'} = 0, \tag{12a}$$
$$(\Box + m^2)A^{\nu\nu}(x) + \frac{\mathrm{i}}{2}(+m^2)(A^{\nu\nu'}(x) - A^{\nu'\nu}(x))\Sigma_{\nu\nu'} + 2m\partial_\mu A^{\nu\nu}(x)\gamma_\mu + 2m\varepsilon_{\mu\nu\nu'\lambda}\partial_\mu A^{\nu\nu'}(x)\gamma_5\gamma_\lambda = 0\,. \tag{12b}$$

By the independence of the γ matrices, (12a) is equivalent to (9a), but (12b) yields new conditions. The first two terms of (12b) give

$$(\Box + m^2)A^{\nu\nu}(x) = 0, \tag{13a}$$
$$(\Box + m^2)(A^{\nu\nu'}(x) - A^{\nu'\nu}(x)) = 0. \tag{13b}$$

On the other hand, (9a) can be rewritten as

$$(\Box - m^2)A^{\nu\nu}(x) = 0, \tag{14a}$$
$$(\Box - m^2)(A^{\nu\nu'}(x) - A^{\nu'\nu}(x)) = 0. \tag{14b}$$

Combining these four equations yields the following two conditions:

$$A^{\nu\nu}(x) = 0, \quad A^{\nu\nu'}(x) = A^{\nu'\nu}(x). \tag{15}$$

The last equation of (15) indicates that $A^{\nu\nu'}(x)$ is symmetric in ν and ν'. From (15), it is easy to see that the last two terms of (12b) vanish. Similarly, by further using (15), (11c) gives $(\Box - m^2)A^{\nu\nu'}(x)\gamma_\nu = 0$, which is again equivalent to (9a). Collecting all the results above, we obtain

$$\Psi_{\alpha\beta\delta\tau}(x) = (\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta} \times (\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\, A^{\nu_1\nu_2}(x)\,, \tag{16}$$

with $A^{\nu_1\nu_2}(x)$ the second rank tensor fields that satisfy the following K–G equations:

$$(\Box - m^2)A^{\nu_1\nu_2}(x) = 0, \tag{17a}$$
$$\partial_{\nu_1}A^{\nu_1\nu_2}(x) = 0, \qquad \partial_{\nu_2}A^{\nu_1\nu_2}(x) = 0, \tag{17b}$$
$$A^{\nu\nu}(x) = 0, \tag{17c}$$
$$A^{\nu_1\nu_2}(x) = A^{\nu_2\nu_1}(x). \tag{17d}$$

2.1.3 Spin 3

The B–W equation for spin 3 reads

$$(\partial\!\!\!/ + m)_{\alpha_1\alpha_1'}\Psi_{\alpha_1'\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x) = 0, \tag{18a}$$
$$(\partial\!\!\!/ + m)_{\beta_1\beta_1'}\Psi_{\alpha_1\beta_1'\alpha_2\beta_2\alpha_3\beta_3}(x) = 0, \tag{18b}$$
$$(\partial\!\!\!/ + m)_{\alpha_2\alpha_2'}\Psi_{\alpha_1\beta_1\alpha_2'\beta_2\alpha_3\beta_3}(x) = 0, \tag{18c}$$
$$(\partial\!\!\!/ + m)_{\beta_2\beta_2'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2'\alpha_3\beta_3}(x) = 0, \tag{18d}$$
$$(\partial\!\!\!/ + m)_{\alpha_3\alpha_3'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3'\beta_3}(x) = 0, \tag{18e}$$
$$(\partial\!\!\!/ + m)_{\beta_3\beta_3'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3'}(x) = 0, \tag{18f}$$

where $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x)$ is a completely symmetric multispinor of rank 6. In exactly the same way as used in the case of spin 2, (18a)–(18d) result in

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x) = (\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha_1\beta_1} \times (\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}\, A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x), \tag{19}$$

with $A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x)$ satisfying the equations below:

$$(\Box - m^2)A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x) = 0, \tag{20a}$$
$$\partial_{\nu_1}A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x) = 0, \quad \partial_{\nu_2}A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x) = 0, \tag{20b}$$
$$A^{\nu\nu}_{\alpha_3\beta_3}(x) = 0, \tag{20c}$$
$$A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x) = A^{\nu_2\nu_1}_{\alpha_3\beta_3}(x). \tag{20d}$$

Substituting (19) into (18e) and (18f), and with the aid of the independence of the matrices $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$, we have

$$(\partial\!\!\!/ + m)_{\alpha_3\alpha_3'}A^{\nu_1\nu_2}_{\alpha_3'\beta_3}(x) = 0, \quad (\partial\!\!\!/ + m)_{\beta_3\beta_3'}A^{\nu_1\nu_2}_{\alpha_3\beta_3'}(x) = 0. \tag{21}$$

Again, they are similar to (1a) and (1b) as far as the the spinor indices α_3 and β_3 are concerned; furthermore, $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x)$ is symmetric in α_3 and β_3, thus we have [refer to (2) and (3a) and (3b)]

$$A^{\nu_2\nu_2}_{\alpha_3\beta_3}(x) = (\mathrm{i}m\gamma_{\nu_3}C + \Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3}\, A^{\nu_1\nu_2\nu_3}(x), \tag{22}$$

with $A^{\nu_1\nu_2\nu_3}(x)$ satisfying the following equations:

$$(\Box - m^2)A^{\nu_1\nu_2\nu_3}(x) = 0 \quad (\nu_1,\nu_2,\nu_3 = 1,2,3,4), \tag{23a}$$
$$\partial_{\nu_3}A^{\nu_1\nu_2\nu_3}(x) = 0. \tag{23b}$$

Substituting (22) into (20b)–(20d), and utilizing the independence of the matrices $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$, it is found that

$$\partial_{\nu_1}A^{\nu_1\nu_2\nu_3}(x) = 0, \quad \partial_{\nu_2}A^{\nu_1\nu_2\nu_3}(x) = 0, \tag{23c}$$
$$A^{\nu\nu\nu_3}(x) = 0, \tag{23d}$$
$$A^{\nu_1\nu_2\nu_3}(x) = A^{\nu_2\nu_1\nu_3}(x). \tag{23e}$$

Combining (22) and (19) yields

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x) = \prod_{i=1}^{3}(\mathrm{i}m\gamma_{\nu_i}C + \Sigma_{\mu_i\nu_i}C\partial_{\mu_i})_{\alpha_i\beta_i} \times A^{\nu_1\nu_2\nu_3}(x). \tag{23f}$$

The right hand side of this expression is symmetric both in $\alpha_1\beta_1\alpha_2\beta_2$, and in α_3 and β_3; in order to gurantee that this expression is completely symmetric in all the six indices $\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3$, we further require that the right hand side of (23f) is also symmetric in $\alpha_2\beta_2\alpha_3\beta_3$ and this is true if the contraction of $(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}(\mathrm{i}m\gamma_{\nu_3}C + \Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3} \times A^{\nu_1\nu_2\nu_3}(x)$, with the three independent antisymmetric *Dirac* matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ where the indices β_2 and α_3 are concerned vanish, namely

$$(\mathrm{i}m\gamma_{\nu_2}C{+}\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}(\mathrm{i}m\gamma_{\nu_3}C{+}\Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3} \times A^{\nu_1\nu_2\nu_3}(x)(C^{-1})_{\beta_2\alpha_3} = 0,$$
$$(\mathrm{i}m\gamma_{\nu_2}C{+}\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}(\mathrm{i}m\gamma_{\nu_3}C{+}\Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3} \times A^{\nu_1\nu_2\nu_3}(x)(C^{-1}\gamma_5)_{\beta_2\alpha_3} = 0,$$
$$(\mathrm{i}m\gamma_{\nu_2}C{+}\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}(\mathrm{i}m\gamma_{\nu_3}C{+}\Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3} \times A^{\nu_1\nu_2\nu_3}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\beta_2\alpha_3} = 0.$$

These equations are analogous to (11a)–(11c) and thus can be expanded in a similar way. The results are that only the second equation yields two new conditions

$$A^{\nu_1\nu\nu}(x) = 0, \quad A^{\nu_1\nu_2\nu_3}(x) = A^{\nu_1\nu_3\nu_2}(x). \tag{23g}$$

Thus we have for spin 3

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x) = \prod_{i=1}^{3}(\mathrm{i}m\gamma_{\nu_i}C + \Sigma_{\mu_i\nu_i}C\partial_{\mu_i})_{\alpha_i\beta_i} \times A^{\nu_1\nu_2\nu_3}(x), \tag{24}$$

with $A^{\nu_1\nu_2\nu_3}(x)$the third rank tensor fields that satisfy the following K–G equations:

$$(\Box - m^2)A^{\nu_1\nu_2\nu_3}(x) = 0, \quad (\nu_1,\nu_2,\nu_3{=}1,2,3,4), \tag{25a}$$
$$\partial_{\nu_1}A^{\nu_1\nu_2\nu_3}(x) = 0, \quad \partial_{\nu_2}A^{\nu_1\nu_2\nu_3}(x) = 0, \quad \partial_{\nu_3}A^{\nu_1\nu_2\nu_3}(x) = 0, \tag{25b}$$
$$A^{\nu\nu\nu_3}(x) = 0, \quad A^{\nu_1\nu\nu}(x) = 0, \tag{25c}$$
$$A^{\nu_1\nu_2\nu_3}(x) = A^{\nu_2\nu_1\nu_3}(x), \quad A^{\nu_1\nu_2\nu_3}(x) = A^{\nu_1\nu_3\nu_2}(x). \tag{25d}$$

2.1.4 Arbitrary integral spin n

The B–W equation for spin n reads

$$(\partial\!\!\!/ + m)_{\alpha_1\alpha_1'}\Psi_{\alpha_1'\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}(x) = 0, \tag{26a}$$

$$\cdots$$

$$(\not\partial + m)_{\beta_n\beta_n'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n'}(x) = 0. \tag{26b}$$

Extending the procedure used to deal with the B–W equation for spin 1, 2 and 3, it is not difficult to find, by the induction method, that the B–W wave functions for spin n take the form

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}(x) = \prod_{i=1}^{n}(\mathrm{i}m\gamma_{\nu_i}C + \Sigma_{\mu_i\nu_i}C\partial_{\mu_i})_{\alpha_i\beta_i} \times A^{\nu_1\nu_2\cdots\nu_n}(x), \tag{27}$$

with $A^{\nu_1\nu_2\cdots\nu_n}(x)$ satisfying the following K–G equations:

$$(\Box - m^2)A^{\nu_1\nu_2\cdots\nu_n}(x) = 0 \quad (\nu_1, \nu_2, \cdots, \nu_n = 1, 2, 3, 4), \tag{28a}$$

$$\partial_{\nu_i}A^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x) = 0 \quad (i = 1, 2, \cdots, n), \tag{28b}$$

$$A^{\cdots\nu\nu\cdots}(x) = 0, \tag{28c}$$

$$A^{\cdots\nu_i\nu_{i+1}\cdots}(x) = A^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i = 1, 2, \cdots, n-1). \tag{28d}$$

2.2 The B–W and R–S equations for half-integral spin

2.2.1 Spin 3/2

The B–W equation for spin 3/2 are

$$(\not\partial + m)_{\alpha\alpha'}\Psi_{\alpha'\beta\rho}(x) = 0, \tag{29a}$$

$$(\not\partial + m)_{\beta\beta'}\Psi_{\alpha\beta'\rho}(x) = 0, \tag{29b}$$

$$(\not\partial + m)_{\rho\rho'}\Psi_{\alpha\beta\rho'}(x) = 0, \tag{29c}$$

where $\Psi_{\alpha\beta\rho}(x)$ is a completely symmetric multispinor of rank 3 and can be expressed as [13]

$$\Psi_{\alpha\beta\rho}(x) = (\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu}C\partial_\mu)_{\alpha\beta}\Psi_\rho^\nu(x), \tag{30}$$

with $\Psi_\rho^\nu(x)$ the vector-spinor satisfying the R–S equations (the spinor index ρ is to be suppressed)

$$(\Box - m^2)\Psi^\nu(x) = 0, \tag{31a}$$

$$\partial_\nu\Psi^\nu(x) = 0, \tag{31b}$$

$$(\not\partial + m)\Psi^\nu(x) = 0, \tag{31c}$$

$$\gamma_\nu\Psi^\nu(x) = 0. \tag{31d}$$

2.2.2 Spin 5/2

The B–W equation for 5/2 spin reads

$$(\not\partial + m)_{\alpha\alpha'}\Psi_{\alpha'\beta\delta\tau\rho}(x) = 0, \tag{32a}$$

$$(\not\partial + m)_{\beta\beta'}\Psi_{\alpha\beta'\delta\tau\rho}(x) = 0, \tag{32b}$$

$$(\not\partial + m)_{\delta\delta'}\Psi_{\alpha\beta\delta'\tau\rho}(x) = 0, \tag{32c}$$

$$(\not\partial + m)_{\tau\tau'}\Psi_{\alpha\beta\delta\tau'\rho}(x) = 0, \tag{32d}$$

$$(\not\partial + m)_{\rho\rho'}\Psi_{\alpha\beta\delta\tau\rho'}(x) = 0, \tag{32e}$$

where $\Psi_{\alpha\beta\delta\tau\rho}(x)$ is a completely symmetric multispinor of rank 5. Comparing (32a)–(32d) with (4a)–(4d), it is easy to see that they satisfy the same Dirac equation with respect to the indices $\alpha\beta\delta\tau$; furthermore, $\Psi_{\alpha\beta\delta\tau\rho}(x)$ is symmetric in $\alpha\beta\delta\tau$; therefore, by using exactly the same procedure which results in (16) and (17a)–(17d), we obtain

$$\Psi_{\alpha\beta\delta\tau\rho}(x) = (\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta} \times (\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\Psi_\rho^{\nu_1\nu_2}(x), \tag{33}$$

with $\Psi_\rho^{\nu_1\nu_2}(x)$ satisfying the equations below:

$$(\Box - m^2)\Psi_\rho^{\nu_1\nu_2}(x) = 0, \tag{34a}$$

$$\partial_{\nu_1}\Psi_\rho^{\nu_1\nu_2}(x) = 0, \quad \partial_{\nu_2}\Psi_\rho^{\nu_1\nu_2}(x) = 0, \tag{34b}$$

$$\Psi_\rho^{\nu\nu}(x) = 0, \tag{34c}$$

$$\Psi_\rho^{\nu_1\nu_2}(x) = \Psi_\rho^{\nu_2\nu_1}(x). \tag{34d}$$

Since $\Psi_\rho^{\nu_1\nu_2}(x)$ also satisfies the Dirac equation with respect to the index ρ, substituting (33) into (32e) yields

$$(\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta}(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau} \times(\not\partial + m)_{\rho\rho'}\Psi_{\rho'}^{\nu_1\nu_2}(x) = 0;$$

by virtue of the independence of the γ matrices, this equation gives

$$(\not\partial + m)_{\rho\rho'}\Psi_{\rho'}^{\nu_1\nu_2}(x) = 0. \tag{34e}$$

On the other hand, the right hand side of (33) is symmetric in the indices $\alpha\beta\delta\tau$; in order to make sure that this expression is completely symmetric in all the five indices $\alpha\beta\delta\tau\rho$, we further require that the right hand side of (33) is also symmetric in the indices $\delta\tau\rho$. The condition for this requirement is that the contraction of the part of $(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\Psi_\rho^{\nu_1\nu_2}(x)$ in (33) with the three independent antisymmetric *Dirac* matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ where the indices τ and ρ are concerned vanish:

$$(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\Psi_\rho^{\nu_1\nu_2}(x)(C^{-1})_{\tau\rho} = 0, \tag{35a}$$

$$(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\Psi_\rho^{\nu_1\nu_2}(x)(C^{-1}\gamma_5)_{\tau\rho} = 0, \tag{35b}$$

$$(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\Psi_\rho^{\nu_1\nu_2}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\tau\rho} = 0. \tag{35c}$$

Expanding (35a), (35b) and (35c) and using (34b)–(34e), with the index ρ being suppressed, we have

$$\gamma_{\nu_2}(\not\partial + m)\Psi^{\nu_1\nu_2}(x) = 0, \tag{36a}$$

$$-2m\gamma_\nu\Psi^{\nu\nu_2}(x) + \gamma_{\nu_2}(\not\partial + m)\Psi^{\nu_1\nu_2}(x) = 0, \tag{36b}$$

$$(\not\partial + m)\Psi^{\nu_1\nu_2}(x) = 0; \tag{36c}$$

both (36a) and (36c) are equivalent to (34e), and the last term of (36b) vanishes while the first term gives

$$\gamma_\nu\Psi^{\nu\nu_2}(x) = 0. \tag{34f}$$

Combining all the results above, we obtain

$$\Psi_{\alpha\beta\delta\tau\rho}(x) = (\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta} \times (\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\Psi^{\nu_1\nu_2}_{\rho}(x), \quad (37)$$

with $\Psi^{\nu_1\nu_2}_{\rho}(x)$ a second rank tensor-spinor satisfying the following R–S equations (with the spinor index ρ being suppressed):

$$(\Box - m^2)\Psi^{\nu_1\nu_2}(x) = 0, \quad (38a)$$

$$\partial_{\nu_1}\Psi^{\nu_1\nu_2}(x) = 0, \quad \partial_{\nu_2}\Psi^{\nu_1\nu_2}(x) = 0, \quad (38b)$$

$$\Psi^{\nu\nu}(x) = 0, \quad (38c)$$

$$\Psi^{\nu_1\nu_2}(x) = \Psi^{\nu_2\nu_1}(x), \quad (38d)$$

$$(\partial\!\!\!/ + m)\Psi^{\nu_1\nu_2}(x) = 0, \quad (38e)$$

$$\gamma_\nu\Psi^{\nu\nu_2}(x) = 0. \quad (38f)$$

2.2.3 Arbitrary half-integral spin

The B–W equation for spin $n + 1/2$ are

$$(\partial\!\!\!/ + m)_{\alpha_1\alpha_1'}\Psi_{\alpha_1'\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = 0, \quad (39a)$$

$$\cdots$$

$$(\partial\!\!\!/ + m)_{\beta_n\beta_n'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n'\rho}(x) = 0, \quad (39b)$$

$$(\partial\!\!\!/ + m)_{\rho\rho'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho'}(x) = 0. \quad (39c)$$

Extending the procedure used to deal with the B–W equation for spin 1/2, 3/2 and 5/2, it is not difficult to find, by the induction method, that the B–W wave functions for spin $n + 1/2$ take the form

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = \prod_{j=1}^{n}\left(\mathrm{i}m\gamma_{\nu_j}C + \Sigma_{\mu_j\nu_j}C\partial_{\mu_j}\right)_{\alpha_j\beta_j} \times\Psi^{\nu_1\nu_2\cdots\nu_n}_{\rho}(x), \quad (40)$$

with $\Psi^{\nu_1\nu_2\cdots\nu_n}_{\rho}(x)$ a rank n tensor-spinor satisfying the following R–S equations (with the spinor index ρ being suppressed):

$$(\Box - m^2)\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad (41a)$$

$$\partial_{\nu_i}\Psi^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x) = 0 \quad (i = 1, 2, \cdots, n), \quad (41b)$$

$$\Psi^{\cdots\nu\nu\cdots}(x) = 0, \quad (41c)$$

$$\Psi^{\cdots\nu_i\nu_{i+1}\cdots}(x) = \Psi^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i = 1, 2, \cdots, n-1), \quad (41d)$$

$$(\partial\!\!\!/ + m)\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad (41e)$$

$$\gamma_\nu\Psi^{\nu\nu_2\nu_3\cdots\nu_n}(x) = 0. \quad (41f)$$

3 Solution to K–G equations for arbitrary integral spins

3.1 Spin 2

In order to solve the K–G equations for spin 2, namely (17a)–(17d), we begin with $A^{\nu_1\nu_2}(x)$ being expanded in plane waves:

$$A^{\nu_1\nu_2}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}kx}A^{\nu_1\nu_2}(k). \quad (42)$$

Substituting this into (17a) yields

$$(k^2 + m^2)A^{\nu_1\nu_2}(k) = 0. \quad (43)$$

By virtue of $x\delta(x) = 0$, $A^{\nu_1\nu_2}(k)$ can be written as

$$A^{\nu_1\nu_2}(k) = \delta(k^2 + m^2)B^{\nu_1\nu_2}(k) = \frac{1}{2\varpi}[\delta(\varpi - k_0) + \delta(\varpi + k_0)]B^{\nu_1\nu_2}(k), \quad (44)$$

where $k = (\vec{k}, \mathrm{i}\varpi)$ and $\varpi = (\vec{k}^2 + m^2)^{1/2}$. Inserting (44) in (42) and integrating over k_0 gives

$$A^{\nu_1\nu_2}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^4}\frac{1}{2\varpi} \times [\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}-\mathrm{i}\varpi t}B^{\nu_1\nu_2}(\vec{k}, \varpi) + \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}+\mathrm{i}\varpi t}B^{\nu_1\nu_2}(-\vec{k}, -\varpi)], \quad (45)$$

or in discrete form with a simplified notation

$$A^{\nu_1\nu_2}(x) = \sum_{\vec{k}}\frac{1}{\sqrt{2\varpi V}}[a^{\nu_1\nu_2}(\vec{k})\mathrm{e}^{\mathrm{i}kx} + b^{\nu_1\nu_2}(\vec{k})\mathrm{e}^{-\mathrm{i}kx}], \quad (46)$$

where $a^{\nu_1\nu_2}(\vec{k})$ and $b^{\nu_1\nu_2}(\vec{k})$ are corresponding to positive and negative energy solutions respectively. Substituting (46) into (17b)–(17d), we obtain the equations in momentum representation:

$$\begin{cases} k_{\nu_1}a^{\nu_1\nu_2}(\vec{k}) = 0, \\ k_{\nu_1}b^{\nu_1\nu_2}(\vec{k}) = 0, \end{cases} \quad \begin{cases} k_{\nu_2}a^{\nu_1\nu_2}(\vec{k}) = 0, \\ k_{\nu_2}b^{\nu_1\nu_2}(\vec{k}) = 0, \end{cases} \quad (47a)$$

$$a^{\nu\nu}(\vec{k}) = 0, \qquad b^{\nu\nu}(\vec{k}) = 0, \quad (47b)$$

$$a^{\nu_1\nu_2}(\vec{k}) = a^{\nu_2\nu_1}(\vec{k}), \qquad b^{\nu_1\nu_2}(\vec{k}) = b^{\nu_2\nu_1}(\vec{k}). \quad (47c)$$

Utilizing $k_\mu e^{\mu}_{\lambda}(\vec{k}) = 0$, the solution to (47a) could be expressed as

$$a^{\nu_1\nu_2}(\vec{k}) = e^{\nu_1}_{\lambda_1}(\vec{k})e^{\nu_2}_{\lambda_2}(\vec{k})a_{\lambda_1\lambda_2}(\vec{k}), \quad (\lambda_1, \lambda_2 = 1, 0, -1), \quad (48a)$$

$$b^{\nu_1\nu_2}(\vec{k}) = \bar{e}^{\nu_1}_{\lambda_1}(\vec{k})\bar{e}^{\nu_2}_{\lambda_2}(\vec{k})b^{+}_{\lambda_1\lambda_2}(\vec{k}), \quad (48b)$$

where $a_{\lambda_1\lambda_2}(\vec{k})$ and $b^{+}_{\lambda_1\lambda_2}(\vec{k})$ need to be further determined, while $e^{\nu}_{\lambda}(\vec{k})$ are the eigenstates of the helicity operator $\vec{S}\cdot(\vec{k}/|\vec{k}|)$ with eigenvalues $\lambda = 1, 0, -1$, namely

$$e^{\nu}_{\pm1}(\vec{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp\cos\theta\cos\phi + \mathrm{i}\sin\phi \\ \mp\cos\theta\sin\phi - \mathrm{i}\cos\phi \\ \pm\sin\theta \\ 0 \end{pmatrix},$$

$$e^{\nu}_{0}(\vec{k}) = \begin{pmatrix} (\varpi/m)\sin\theta\cos\phi \\ (\varpi/m)\sin\theta\sin\phi \\ (\varpi/m)\cos\theta \\ \mathrm{i}|\vec{k}|/m \end{pmatrix} \quad (49a)$$

(θ, ϕ are the direction angles of $\vec{k}$) and

$$\bar{e}^{\nu}_{\lambda}(\vec{k}) = g_{\nu\mu}\left(e^{\mu}_{\lambda}(\vec{k})\right)^{*}, \quad g_{\nu\mu} = \mathrm{diag}\{1,1,1,-1\}. \quad (49b)$$

In the rest frame, $\vec{k}=0$, $e^{\nu}_{\lambda}(\vec{k})$ is simplified to

$$e^{\nu}_{+1}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-\mathrm{i}\\0\\0\end{pmatrix}, \quad e^{\nu}_{0}(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},$$

$$e^{\nu}_{-1}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-\mathrm{i}\\0\\0\end{pmatrix}, \quad (50)$$

which are the eigenstates of S_z with eigenvalues $1, 0, -1$. $e^{\nu}_{\lambda}(\vec{k})$ and $e^{\nu}_{\lambda}(0)$ can be connected by a Lorentz transformation,

$$e^{\nu}_{\lambda}(\vec{k}) = L^{\nu\nu'}e^{\nu'}_{\lambda}(0) \quad (\lambda = 1, 0, -1), \quad (51)$$

with

$$L = \mathrm{e}^{-\mathrm{i}S_3\phi}\mathrm{e}^{-\mathrm{i}S_2\theta}\mathrm{e}^{\varepsilon K_3}; \quad (52)$$

the explicit matrix form of this Lorentz transformation is given in Appendix A. Substituting (48a) and (48b) into (47b) gives

$$e^{\nu}_{\lambda_1}(\vec{k})e^{\nu}_{\lambda_2}(\vec{k})a_{\lambda_1\lambda_2}(\vec{k}) = 0,$$
$$e^{\nu}_{\lambda_1}(\vec{k})e^{\nu}_{\lambda_2}(\vec{k})b^{+}_{\lambda_1\lambda_2}(\vec{k}) = 0, \quad (53)$$

Utilizing (51) and noticing that $L^{\nu\nu_1}L^{\nu\nu_2} = \delta_{\nu_1\nu_2}$, (53) can be rewritten as

$$e^{\nu}_{\lambda_1}(0)e^{\nu}_{\lambda_2}(0)a_{\lambda_1\lambda_2}(\vec{k}) = 0, \quad (54a)$$
$$e^{\nu}_{\lambda_1}(0)e^{\nu}_{\lambda_2}(0)b^{+}_{\lambda_1\lambda_2}(\vec{k}) = 0. \quad (54b)$$

Now we focus on the solution to (54a); the solution to (54b) will be obtained in the same way. Equation (54a) indicates that $a_{\lambda_1\lambda_2}(\vec{k})$ is related to the two magnetic quantum numbers λ_1 and λ_2 ($\lambda_1, \lambda_2 = 1, 0, -1$). Recalling the Clebsch–Gordan coefficients for the coupling of two spin-1 angular momenta, a general candidate for $a_{\lambda_1\lambda_2}(\vec{k})$ is

$$\begin{aligned} a_{\lambda_1\lambda_2}(\vec{k}) = &\sum_{m}\langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle a_{2m}(\vec{k}) \\ &+ \sum_{m'}\langle 1,\lambda_1;1,\lambda_2|1,1,1,m'\rangle a_{1m'}(\vec{k}) \\ &+ \langle 1,\lambda_1;1,\lambda_2|1,1,0,0\rangle a_{00}(\vec{k}), \\ &(m = 2,1,0,-1,-2; \quad m' = 1,0,-1), \end{aligned} \quad (55)$$

with $\langle S_1,\lambda_1;S_2,\lambda_2|S_1,S_2,S,m\rangle$ the Clebsch–Gordan coefficients for the coupling $\vec{S} = \vec{S}_1 + \vec{S}_2 \quad (S_1 = S_2 = 1, S = 2, 1, 0)$. Let

$$e^{\nu_1\nu_2}_{2m}(0) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(0)e^{\nu_2}_{\lambda_2}(0)\langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle$$
$$(m = 0, \pm 1, \pm 2), \quad (56a)$$

$$e^{\nu_1\nu_2}_{1m'}(0) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(0)e^{\nu_2}_{\lambda_2}(0)\langle 1,\lambda_1;1,\lambda_2|1,1,1,m'\rangle$$
$$(m' = 0, \pm 1), \quad (56b)$$

$$e^{\nu_1\nu_2}_{00}(0) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(0)e^{\nu_2}_{\lambda_2}(0)\langle 1,\lambda_1;1,\lambda_2|1,1,0,0\rangle, \quad (56c)$$

then (54a) takes the form

$$e^{\nu\nu}_{2m}(0)a_{2m}(\vec{k}) + e^{\nu\nu}_{1m'}(0)a_{1m'}(\vec{k}) + e^{\nu\nu}_{00}(0)a_{00}(\vec{k}) = 0. \quad (57)$$

With the aid of (50) and the explicit expression of (56a), (56b) and (56c) listed in Appendix B, we find

$$e^{\nu\nu}_{2m}(0) = 0 \quad (m = 2,1,0,-1,-2), \quad (58)$$
$$e^{\nu\nu}_{1m'}(0) = 0 \quad (m' = 1,0,-1), \quad e^{\nu\nu}_{00}(0) = -\sqrt{3}.$$

Substituting (58) into (57) gives

$$a_{00}(\vec{k}) = 0; \quad (59)$$

thus (55) is simplified to

$$\begin{aligned} a_{\lambda_1\lambda_2}(\vec{k}) = &\sum_{m}\langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle a_{2m}(\vec{k}) \\ &+ \sum_{m'}\langle 1,\lambda_1;1,\lambda_2|1,1,1,m'\rangle a_{1m'}(\vec{k}). \end{aligned} \quad (60a)$$

In exactly the same way, the solution to (54b) has the form

$$\begin{aligned} b^{+}_{\lambda_1\lambda_2}(\vec{k}) = &\sum_{m}\langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle b^{+}_{2m}(\vec{k}) \\ &+ \sum_{m'}\langle 1,\lambda_1;1,\lambda_2|1,1,1,m'\rangle b^{+}_{1m'}(\vec{k}). \end{aligned} \quad (60b)$$

On the other hand, the condition (47c) leads to

$$a_{\lambda_1\lambda_2}(\vec{k}) = a_{\lambda_2\lambda_1}(\vec{k}), \quad b^{+}_{\lambda_1\lambda_2}(\vec{k}) = b^{+}_{\lambda_2\lambda_1}(\vec{k}); \quad (61)$$

however,

$$\langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle = \langle 1,\lambda_2;1,\lambda_1|1,1,2,m\rangle, \quad (62a)$$
$$\langle 1,\lambda_1;1,\lambda_2|1,1,1,m'\rangle = -\langle 1,\lambda_2;1,\lambda_1|1,1,1,m'\rangle; \quad (62b)$$

substituting (60a) and (60b) into (61) and with the help of (62a) and (62b), we have

$$a_{1m'}(\vec{k}) = 0, \quad b^{+}_{1m'}(\vec{k}) = 0; \quad (63)$$

therefore (60a) and (60b) are simplified to

$$a_{\lambda_1\lambda_2}(\vec{k}) = \sum_{m}\langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle a_{2m}(\vec{k}), \quad (64a)$$

$$b^{+}_{\lambda_1\lambda_2}(\vec{k}) = \sum_{m}\langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle b^{+}_{2m}(\vec{k}), \quad (64b)$$

and (48a) and (48b) become correspondingly

$$a^{\nu_1\nu_2}(\vec{k}) = e^{\nu_1\nu_2}_{2m}(\vec{k}) a_{2m}(\vec{k}),$$
$$b^{\nu_1\nu_2}(\vec{k}) = \bar{e}^{\nu_1\nu_2}_{2m}(\vec{k}) b^{+}_{2m}(\vec{k}), \tag{65}$$

with

$$e^{\nu_1\nu_2}_{2m}(\vec{k}) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(\vec{k}) e^{\nu_2}_{\lambda_2}(\vec{k}) \langle 1, \lambda_1; 1, \lambda_2 | 1, 1, 2, m\rangle$$
$$(m = 2, 1, 0, -1, -2), \tag{66a}$$

and

$$\bar{e}^{\nu_1\nu_2}_{2m}(\vec{k}) = g_{\nu_1\mu_1} g_{\nu_1\mu_2} \left(e^{\mu_1\mu_2}_{2m}(\vec{k})\right)^*. \tag{66b}$$

Substituting (65) back into (46), omitting the index 2, we obtain

$$A^{\nu_1\nu_2}(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2\varpi V}} \Big[a_m(\vec{k}) e^{\nu_1\nu_2}_m(\vec{k}) \mathrm{e}^{\mathrm{i}kx}$$
$$+ b^{+}_m(\vec{k}) \bar{e}^{\nu_1\nu_2}_m(k) \mathrm{e}^{-\mathrm{i}kx} \vec{k} \mathrm{e}^{-\mathrm{i}kx} \Big], \tag{67}$$

with (using the Wigner formula for the Clebsch–Gordan coefficients)

$$e^{\nu_1\nu_2}_m(\vec{k}) = \sum_{\lambda_1,\lambda_2=-1}^{1} e^{\nu_1}_{\lambda_1}(\vec{k}) e^{\nu_2}_{\lambda_2}(\vec{k}) \delta(\lambda_1 + \lambda_2, m) \tag{68}$$
$$\times \sqrt{\frac{(2+m)!(2-m)!}{6(1+\lambda_1)!(1-\lambda_1)!(1+\lambda_2)!(1-\lambda_2)!}}.$$

Consulting the Clebsch–Gordan coefficients listed in Appendix B, this expression is in agreement with that derived by Chung [1], (39a)–(39c).

3.2 Spin 3

The K–G equations for spin 3 [(25a)–(25d)] could be rewritten as

$$(\Box - m^2) A^{\nu_1\nu_2\nu_3}(x) = 0, \tag{69a}$$
$$\partial_{\nu_1} A^{\nu_1\nu_2\nu_3}(x) = 0, \quad \partial_{\nu_2} A^{\nu_1\nu_2\nu_3}(x) = 0,$$
$$A^{\nu_1\nu_2\nu_3}(x) = A^{\nu_2\nu_1\nu_3}(x),$$
$$A^{\nu\nu\nu_3}(x) = 0, \tag{69b}$$
$$\partial_{\nu_3} A^{\nu_1\nu_2\nu_3}(x) = 0, \tag{69c}$$
$$A^{\nu_1\nu\nu}(x) = 0, \tag{69d}$$
$$A^{\nu_1\nu_2\nu_3}(x) = A^{\nu_1\nu_3\nu_2}(x). \tag{69e}$$

Utilizing the procedure performed in the above section for spin 2, we can write down from (69a) and (69b)

$$A^{\nu_1\nu_2\nu_3}(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2\varpi V}} \tag{70}$$
$$\times \left[a^{\nu_1\nu_2\nu_3}(\vec{k}) \mathrm{e}^{\mathrm{i}kx} + b^{\nu_1\nu_2\nu_3}(\vec{k}) \mathrm{e}^{-\mathrm{i}kx}\right],$$

with

$$a^{\nu_1\nu_2\nu_3}(\vec{k}) = e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{k}) a^{\nu_3}_{\lambda_{12}}(\vec{k}),$$
$$b^{\nu_1\nu_2\nu_3}(\vec{k}) = \bar{e}^{\nu_1\nu_2}_{\lambda_{12}}(\vec{k}) b^{\nu_3^{+}}_{\lambda_{12}}(\vec{k}), \tag{71a}$$
$$e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{k}) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(\vec{k}) e^{\nu_2}_{\lambda_2}(\vec{k}) \langle 1, \lambda_1; 1, \lambda_2 | 1, 1, 2, \lambda_{12}\rangle,$$
$$(\lambda_{12} = 2, 1, 0, -1, -2). \tag{71b}$$

Substituting (70) into (69c)–(69e) yields

$$k_{\nu_3} a^{\nu_1\nu_2\nu_3}(\vec{k}) = 0, \quad k_{\nu_3} b^{\nu_1\nu_2\nu_3}(\vec{k}) = 0, \tag{72a}$$
$$a^{\nu_1\nu\nu}(\vec{k}) = 0, \quad b^{\nu_1\nu\nu}(\vec{k}) = 0, \tag{72b}$$
$$a^{\nu_1\nu_2\nu_3}(\vec{k}) = a^{\nu_1\nu_3\nu_2}(\vec{k}),$$
$$b^{\nu_1\nu_2\nu_3}(\vec{k}) = b^{\nu_1\nu_3\nu_2}(\vec{k}). \tag{72c}$$

The solution to (72a) could be expressed as

$$a^{\nu_1\nu_2\nu_3}(\vec{k}) = e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{k}) e^{\nu_3}_{\lambda_3}(\vec{k}) a_{\lambda_{12}\lambda_3}(\vec{k}),$$
$$b^{\nu_1\nu_2\nu_3}(\vec{k}) = \bar{e}^{\nu_1\nu_2}_{\lambda_{12}}(\vec{k}) \bar{e}^{\nu_3}_{\lambda_3}(\vec{k}) b^{+}_{\lambda_{12}\lambda_3}(\vec{k}). \tag{73}$$

Substituting (73) into (72b) results in

$$e^{\nu_1\nu}_{\lambda_{12}}(\vec{k}) e^{\nu}_{\lambda_3}(\vec{k}) a_{\lambda_{12}\lambda_3}(\vec{k}) = 0,$$
$$e^{\nu_1\nu}_{\lambda_{12}}(\vec{k}) e^{\nu}_{\lambda_3}(\vec{k}) b^{+}_{\lambda_{12}\lambda_3}(\vec{k}) = 0, \tag{74}$$

which can be rewritten with the aid of (51), (71b) and $\Gamma^{\nu\nu_1}\Gamma^{\nu\nu_2} = \delta_{\nu_1\nu_2}$ as

$$e^{\nu_1\nu}_{\lambda_{12}}(0) e^{\nu}_{\lambda_3}(0) a_{\lambda_{12}\lambda_3}(\vec{k}) = 0, \tag{75a}$$
$$e^{\nu_1\nu}_{\lambda_{12}}(0) e^{\nu}_{\lambda_3}(0) b^{+}_{\lambda_{12}\lambda_3}(\vec{k}) = 0. \tag{75b}$$

Equation (75a) indicates that $a_{\lambda_{12}\lambda_3}(\vec{k})$ is related to the two magnetic quantum numbers λ_{12} and λ_3 ($\lambda_{12} = 2, 1, 0, -1, -2$, $\lambda_3 = 1, 0, -1$). Recalling the Clebsch–Gordan coefficients for the coupling of two angular momenta with spin 2 and 1 respectively, a general candidate for $a_{\lambda_{12}\lambda_3}(\vec{k})$ is

$$a_{\lambda_{12}\lambda_3}(\vec{k}) = \sum_m \langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 3, m\rangle a_{3m}(\vec{k})$$
$$+ \sum_{m'} \langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 2, m'\rangle a_{2m'}(\vec{k})$$
$$+ \sum_{m''} \langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 1, m''\rangle a_{1m''}(\vec{k}) \tag{76}$$
$$(m = 3, 2, 1, 0, -1, -2, -3;$$
$$m' = 2, 1, 0, -1, -2;$$
$$m'' = 1, 0, -1).$$

Let

$$e^{\nu_1\nu_2\nu_3}_{3m}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0) e^{\nu_3}_{\lambda_3}(0) \langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 3, m\rangle, \tag{77a}$$

$$e^{\nu_1\nu_2\nu_3}_{2m'}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0)e^{\nu_3}_{\lambda_3}(0)\langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle, \tag{77b}$$

$$e^{\nu_1\nu_2\nu_3}_{1m''}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0)e^{\nu_3}_{\lambda_3}(0)\langle 2,\lambda_{12};1,\lambda_3|2,1,1,m''\rangle, \tag{77c}$$

then (75a) takes the form

$$e^{\nu_1\nu\nu}_{3m}(0)a_{3m}(\vec{k})+e^{\nu_1\nu\nu}_{2m'}(0)a_{2m'}(\vec{k})+e^{\nu_1\nu\nu}_{1m''}(0)a_{1m''}(\vec{k}) = 0. \tag{78}$$

With the help of the explicit form of (77a) and (77b) which are listed in Appendix C and the following relations derived from (50):

$$\begin{aligned} e^{\nu}_1(0)e^{\nu}_1(0) &= e^{\nu}_{-1}(0)e^{\nu}_{-1}(0) = e^{\nu}_0(0)e^{\nu}_1(0) \\ &= e^{\nu}_1(0)e^{\nu}_0(0) = e^{\nu}_{-1}(0)e^{\nu}_0(0) \\ &= e^{\nu}_0(0)e^{\nu}_{-1}(0) = 0, \\ e^{\nu}_1(0)e^{\nu}_{-1}(0) &= e^{\nu}_{-1}(0)e^{\nu}_1(0) = -1, \\ e^{\nu}_0(0)e^{\nu}_0(0) &= 1, \end{aligned}$$

we find

$$\begin{aligned} &e^{\nu_1\nu\nu}_{3m}(0) = 0 \quad (m = 3,2,1,0,-1,-2,-3), \\ &e^{\nu_1\nu\nu}_{2m'}(0) = 0 \quad (m' = 2,1,0,-1,-2), \end{aligned} \tag{79a}$$

$$e^{\nu_1\nu\nu}_{1m''}(0) = -\sqrt{\frac{5}{3}}e^{\nu_1}_{m''}(0) \neq 0 \quad (m'' = 1,0,-1). \tag{79b}$$

Substituting (79a) and (79b) into (78) yields

$$a_{1m''}(\vec{k}) = 0 \quad (m'' = 1,0,-1); \tag{80}$$

thus (76) is simplified to

$$\begin{aligned} a_{\lambda_{12}\lambda_3}(\vec{k}) &= \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle a_{3m}(\vec{k}) \\ &+ \sum_{m'} \langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle a_{2m'}(\vec{k}). \end{aligned} \tag{81a}$$

Similarly, (75b) leads to

$$\begin{aligned} b^+_{\lambda_{12}\lambda_3}(\vec{k}) &= \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle b^+_{3m}(\vec{k}) \\ &+ \sum_{m'} \langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle b^+_{2m'}(\vec{k}), \end{aligned} \tag{81b}$$

On the other hand, the condition (72c) requires

$$a_{\lambda_{12}\lambda_3}(\vec{k}) = a_{\lambda_3\lambda_{12}}(\vec{k}), b^+_{\lambda_{12}\lambda_3}(\vec{k}) = b^+_{\lambda_3\lambda_{12}}(\vec{k}); \tag{82}$$

however,

$$\langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle = \langle 1,\lambda_3;2,\lambda_{12}|1,2,3,m\rangle, \tag{83a}$$
$$\langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle = -\langle 1,\lambda_3;2,\lambda_{12}|1,2,2,m'\rangle. \tag{83b}$$

Substituting (81a) and (81b) into (82) and with the aid of (83a) and (83b), we have

$$a_{2m'}(\vec{k}) = 0, \quad b^+_{2m'}(\vec{k}) = 0 \quad (m' = 2,1,0,-1,-2). \tag{84}$$

Therefore, (81a) and (81b) are simplified to

$$a_{\lambda_{12}\lambda_3}(\vec{k}) = \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle a_{3m}(\vec{k}), \tag{85a}$$

$$b_{\lambda_{12}\lambda_3}(\vec{k}) = \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle b^+_{3m}(\vec{k}), \tag{85b}$$

and (73) becomes correspondingly

$$\begin{aligned} a^{\nu_1\nu_2\nu_3}(\vec{k}) &= e^{\nu_1\nu_2\nu_3}_{3m}(\vec{k})a_{3m}(\vec{k}), \\ b^{\nu_1\nu_2\nu_3}(\vec{k}) &= \bar{e}^{\nu_1\nu_2\nu_3}_{3m}(\vec{k})b^+_{3m}(\vec{k}), \end{aligned} \tag{86a}$$

with

$$\begin{aligned} e^{\nu_1\nu_2\nu_3}_{3m}(\vec{k}) &= \sum_{\lambda_{12}\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{k})e^{\nu_3}_{\lambda_3}(\vec{k}) \\ &\times \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle, \end{aligned} \tag{86b}$$

$$\bar{e}^{\nu_1\nu_2\nu_3}_{3m}(\vec{k}) = g_{\nu_1\mu_1}g_{\nu_2\mu_2}g_{\nu_3\mu_3}\left(e^{\mu_1\mu_2\mu_3}_{3m}(\vec{k})\right)^* \quad (m = 3,2,1,0,-1,-2,-3), \tag{86c}$$

and the wave function for spin 3 takes the final form (omitting the index 3)

$$\begin{aligned} A^{\nu_1\nu_2\nu_3}(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2\varpi V}}&[a_m(\vec{k})e^{\nu_1\nu_2\nu_3}_m(\vec{k})\mathrm{e}^{\mathrm{i}kx} \\ &+ b^+_m(\vec{k})\bar{e}^{\nu_1\nu_2\nu_3}_m(\vec{k})\mathrm{e}^{-\mathrm{i}kx}], \end{aligned} \tag{87}$$

where (using the Wigner formula for the Clebsch–Gordan coefficients)

$$\begin{aligned} &e^{\nu_1\nu_2\nu_3}_m(\vec{k}) \\ &= \sum_{\lambda_1,\lambda_2\lambda_3=-1}^{1} e^{\nu_1}_{\lambda_1}(\vec{k})e^{\nu_2}_{\lambda_2}(\vec{k})e^{\nu_3}_{\lambda_3}(\vec{k})\delta(\lambda_1+\lambda_2+\lambda_3,m) \\ &\times \sqrt{\frac{(3+m)!(3-m)!}{90\prod_{i=1}^{3}(1+\lambda_i)!(1-\lambda_i)!}}. \end{aligned} \tag{88}$$

Comparing the Clebsch–Gordan coefficients listed in Appendix C for this expression with those derived by Chung [1], (40), one finds they are in agreement with each other.

3.3 Spin n

As an extension of the results for spin 2 and 3, the solution to the K–G equations for integral spin n, (28a)–(28d), could be expressed as

$$\begin{aligned} A^{\nu_1\nu_2\cdots\nu_n}(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2\varpi V}}&[a_m(\vec{k})e^{\nu_1\nu_2\ldots\nu_n}_m\vec{k}\mathrm{e}^{\mathrm{i}kx} \\ &+ b^+_m(\vec{k})\bar{e}^{\nu_1\nu_2\ldots\nu_n}_m\vec{k}\mathrm{e}^{-\mathrm{i}kx}], \end{aligned} \tag{89}$$

where $m = 0, \pm 1, \pm 2, \cdots \pm n$,

$$e^{\nu_1\nu_2\cdots\nu_n}_m(\vec{k}) = \sum_{\lambda_i=-1}^{1}\prod_{i=1}^{n} e^{\nu_i}_{\lambda_i}(\vec{k})\prod_{i=1}^{n-1}\Big\langle i,\lambda_1+\lambda_2$$

$$+\cdots\lambda_i;1,\lambda_{i+1}|i,1,i+1,\lambda_1+\lambda_2+\cdots\lambda_{i+1}\Big\rangle$$
$$\equiv \sum_{\lambda_1\lambda_2\cdots\lambda_n=-1}^{1}\delta(\lambda_1+\lambda_2+\cdots\lambda_n,m)$$
$$\times\sqrt{\frac{2^n(n+m)!(n-m)!}{(2n)!\prod_{i=1}^{n}(1+\lambda_i)!(1-\lambda_i)!}}\prod_{i=1}^{n}e^{\nu_i}_{\lambda_i}(\vec{k}), \tag{90a}$$
$$\bar{e}^{\nu_1\nu_2\ldots\nu_n}_m(\vec{k})=g_{\nu_1\mu_1}g_{\nu_2\mu_2}\cdots g_{\nu_n\mu_n}\left(e^{\mu_1\mu_2\ldots\mu_n}_m(\vec{k})\right)^*. \tag{90b}$$

This conclusion could be straightforwardly proved by the reduction method. We will not express the details here but only outline the main points. With spin increasing from $n-1$ to n by one unit, three new subsidiary conditions are added to the K–G equations for spin n. These are the Lorentz condition $\partial_{\nu_n}A^{\nu_1\nu_2\cdots\nu_n}(x)=0$, the traceless condition $A^{\cdots\nu_{n-1}\nu_n}(x)=0$, and the symmetric condition $A^{\cdots\nu_{n-1}\nu_n}(x)=A^{\cdots\nu_n\nu_{n-1}}(x)$. The Lorentz condition leads to a coupling between spin-$n-1$ wave functions and spin-1 wave functions, which results in three classes of different total spin wave functions, namely wave functions for spin n, $n-1$ and $n-2$. The traceless condition removes the spin-$n-2$ wave functions and the symmetric condition removes the spin-$n-1$ wave functions. Thus only the spin n wave functions, corresponding to the maximum possible spin, are kept in the final expression of the solution.

Following Chung [1], expression (90a) can be rewritten in a more closed form as

$$e^{\nu_1\nu_2\cdots\nu_n}_m(\vec{k})=\sqrt{\frac{(n+m)!(n-m)!}{(2n)!}}\sum_{m_0}2^{m_0/2} \tag{91}$$
$$\times\sum_P\left\{\prod_{i=1}^{m_+}e^{\nu_i}_1(\vec{k})\prod_{j=m_++1}^{m_++m_0}e^{\nu_j}_0(\vec{k})\prod_{k=m_++m_0+1}^{m_++m_0+m_-}e^{\nu_k}_{-1}(\vec{k})\right\},$$

where $m_\pm$ stands for the numbers of $e^\nu_{\pm1}(\vec{k})$'s and m_0 for $e^\nu_0(\vec{k})$'s, and their values are

$$m_++m_0+m_-=n,\quad m_+-m_-=m,$$
$$m_0=\begin{cases}1,3,5,\cdots,n-m & (\text{for } n-m=\text{odd}),\\ 0,2,4,\cdots,n-m & (\text{for } n-m=\text{even}),\end{cases}$$
$$m_\pm=\frac{1}{2}(n\pm m-m_0).$$

The first sum in (91) goes over the allowed values of m_0 given n and m, while the second sum in (91) represents a summation on the permutations

$$\left\{\prod_{i=1}^{m_+}e^{\nu_i}_1(\vec{k})\prod_{j=m_++1}^{m_++m_0}e^{\nu_j}_0(\vec{k})\prod_{k=m_++m_0+1}^{m_++m_0+m_-}e^{\nu_k}_{-1}(\vec{k})\right\}.$$

The equivalence between (90a) and (91) is clear if one notices

$$\prod_{i=1}^{n}(1+\lambda_i)!(1-\lambda_i)!=2^{m_++m_-}.$$

4 Solution to R–S equation for arbitrary half-integral spin

4.1 Spin 3/2

In this section, the R–S equation for arbitrary half-integral spin is to be solved. For the simpler case of (31a)–(31d), utilizing the familiar wave function for spin 1, we have first from (31a) and (31b)

$$\Psi^\nu(x)=\frac{1}{\sqrt{V}}\sum_{\vec{p}}\left[e^\nu_\lambda(\vec{p})a_\lambda(\vec{p})e^{ipx}+\bar{e}^\nu_\lambda(\vec{p})b^+_\lambda(\vec{p})e^{-ipx}\right], \tag{92}$$

with $p=(\vec{p},iE)$ and $E=(\vec{p}^2+m^2)^{1/2}$. Substituting (92) into (31c) yields

$$(i\not{p}+m)a_\lambda(\vec{p})=0,$$
$$(-i\not{p}+m)b^+_\lambda(\vec{p})=0, \tag{93}$$

which are the well-known Dirac equations for spin 1/2, and their solutions are

$$a_\lambda(\vec{p})=u_r(\vec{p})a_{\lambda,r}(\vec{p}),\quad\left(r=\frac{1}{2},-\frac{1}{2}\right), \tag{94a}$$
$$b^+_\lambda(\vec{p})=v_r(\vec{p})b^+_{\lambda,r}(\vec{p}),$$
$$v_r(\vec{p})=\gamma_2(u_r(\vec{p}))^*, \tag{94b}$$

where $u_r(\vec{p})$ and $v_r(\vec{p})$ are the positive and negative energy spinors respectively and can be expressed as

$$u_r(\vec{p})=\Lambda u_r(0)\sqrt{\frac{m}{E}},\quad v_r(\vec{p})=\Lambda v_r(0)\sqrt{\frac{m}{E}}, \tag{95}$$

with

$$\Lambda=e^{-i\Sigma_3\phi/2}e^{-i\Sigma_2\theta/2}e^{\alpha_3\varepsilon/2}, \tag{96}$$

and

$$u_{1/2}(0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad u_{-1/2}(0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix};$$
$$v_{1/2}(0)=\begin{pmatrix}0\\0\\0\\-1\end{pmatrix},\quad v_{-1/2}(0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \tag{97}$$

which are positive and negative energy spinors in the rest frame. Inserting (94a) and (94b) in (92) yields

$$\Psi^\nu(x)=\frac{1}{\sqrt{V}}\sum_{\vec{p}}\Big[e^\nu_\lambda(\vec{p})u_r(\vec{p})a_{\lambda,r}(\vec{p})e^{ipx}$$
$$+\bar{e}^\nu_\lambda(\vec{p})v_r(\vec{p})b^+_{\lambda,r}(\vec{p})e^{-ipx}\Big]. \tag{98}$$

Substituting (98) into (31d), we have

$$\gamma_\nu e^\nu_\lambda(\vec{p})u_r(\vec{p})a_{\lambda,r}(\vec{p})=0, \tag{99a}$$

$$\gamma_\nu \bar{e}^\nu_\lambda(\vec{p}) v_r(\vec{p}) b^+_{\lambda,r}(\vec{p}) = 0. \tag{99b}$$

Utilizing $\bar{e}^\nu_\lambda(\vec{p}) = g_{\nu\mu}(e^\mu_\lambda(\vec{p}))^*$, $\gamma_2\gamma^*_\nu\gamma_2 = g_{\nu\mu}\gamma_\mu$ and $v_r(\vec{p}) = \gamma_2 u_r(\vec{p})^*$, it is easy to show that (99b) is equivalent to (99a); thus we will focus on (99a). With the aid of (51) and (95) and noticing

$$\Gamma^{\nu\nu_1}\Gamma^{\nu\nu_2} = \delta_{\nu_1\nu_2}, \quad \Lambda^{-1}\gamma_\nu\Lambda = \Gamma^{\nu\nu'}\gamma_{\nu'}, \tag{100}$$

(99a) can be rewritten as

$$\gamma_\nu e^\nu_\lambda(0) u_r(0) a_{\lambda,r}(\vec{p}) = 0. \tag{101}$$

Equation (101) indicates that $a_{\lambda,r}(\vec{p})$ is related to the two magnetic quantum numbers λ and r ($\lambda = 1, 0, -1$; $r = 1/2, -1/2$); recalling the Clebsch–Gordan coefficients for coupling a spin-1 angular momentum to a spin-1/2 one, a general candidate for $a_{\lambda,r}(\vec{p})$ is

$$\begin{aligned} a_{\lambda,r}(\vec{p}) = &\sum_m \left\langle 1,\lambda;\frac{1}{2},r \middle| 1,\frac{1}{2},\frac{3}{2},m \right\rangle a_{3/2,m}(\vec{p}) \\ &+ \sum_{m'} \left\langle 1,\lambda;\frac{1}{2},r \middle| 1,\frac{1}{2},\frac{1}{2},m' \right\rangle a_{1/2,m'}(\vec{p}), \end{aligned} \tag{102}$$

where $m = \pm 1/2, \pm 3/2, m' = \pm 1/2$.

Let

$$U^\nu_{3/2,m}(\vec{p}) = \sum_{\lambda,r} e^\nu_\lambda(\vec{p}) u_r(\vec{p}) \left\langle 1,\lambda;\frac{1}{2},r \middle| 1,\frac{1}{2},\frac{3}{2},m \right\rangle, \tag{103a}$$

$$U^\nu_{1/2,m'}(\vec{p}) = \sum_{\lambda,r} e^\nu_\lambda(\vec{p}) u_r(\vec{p}) \left\langle 1,\lambda;\frac{1}{2},r \middle| 1,\frac{1}{2},\frac{1}{2},m' \right\rangle, \tag{103b}$$

then (101) takes the form

$$\gamma_\nu U^\nu_{3/2,m}(0) a_{3/2,m}(\vec{p}) + \gamma_\nu U^\nu_{1/2,m'}(0) a_{1/2,m'}(\vec{p}) = 0. \tag{104}$$

The explicit expression of (103a) and (103b), after the CG coefficients have been calculated, is

$$\begin{aligned} U^\nu_{3/2,3/2}(\vec{p}) &= e^\nu_{+1}(\vec{p}) u_{1/2}(\vec{p}), \\ U^\nu_{3/2,1/2}(\vec{p}) &= \sqrt{\frac{1}{3}} e^\nu_{+1}(\vec{p}) u_{-1/2}(\vec{p}) + \sqrt{\frac{2}{3}} e^\nu_0(\vec{p}) u_{1/2}(\vec{p}), \\ U^\nu_{3/2,-1/2}(\vec{p}) &= \sqrt{\frac{2}{3}} e^\nu_0(\vec{p}) u_{-1/2}(\vec{p}) + \sqrt{\frac{1}{3}} e^\nu_{-1}(\vec{p}) u_{1/2}(\vec{p}), \\ U^\nu_{3/2,-3/2}(\vec{p}) &= e^\nu_{-1}(\vec{p}) u_{-1/2}(\vec{p}), \\ U^\nu_{1/2,1/2}(\vec{p}) &= \sqrt{\frac{2}{3}} e^\nu_{+1}(\vec{p}) u_{-1/2}(\vec{p}) - \sqrt{\frac{1}{3}} e^\nu_0(\vec{p}) u_{1/2}(\vec{p}), \\ U^\nu_{1/2,-1/2}(\vec{p}) &= \sqrt{\frac{1}{3}} e^\nu_0(\vec{p}) u_{-1/2}(\vec{p}) + \sqrt{\frac{2}{3}} e^\nu_{-1}(\vec{p}) u_{1/2}(\vec{p}). \end{aligned}$$

With the aid of the following relations, which can be derived from (50) and (97):

$$\gamma_\nu e^\nu_{+1}(0) u_{1/2}(0) = 0,$$

$$\gamma_\nu e^\nu_{+1}(0) u_{-1/2}(0) = \mathrm{i}\sqrt{2}\gamma_5 u_{1/2}(0), \tag{105a}$$

$$\begin{aligned} \gamma_\nu e^\nu_0(0) u_{1/2}(0) &= -\mathrm{i}\gamma_5 u_{1/2}(0), \\ \gamma_\nu e^\nu_0(0) u_{-1/2}(0) &= \mathrm{i}\gamma_5 u_{-1/2}(0), \end{aligned} \tag{105b}$$

$$\begin{aligned} \gamma_\nu e^\nu_{-1}(0) u_{1/2}(0) &= -\mathrm{i}\sqrt{2}\gamma_5 u_{-1/2}(0), \\ \gamma_\nu e^\nu_{-1}(0) u_{-1/2}(0) &= 0, \end{aligned} \tag{105c}$$

we find

$$\gamma_\nu U^\nu_{3/2,m}(0) = 0 \quad \left(m = \pm\frac{1}{2}, \pm\frac{3}{2}\right), \tag{106}$$

$$\gamma_\nu U^\nu_{1/2,m'}(0) = \mathrm{i}\sqrt{3}\gamma_5 u_{m'}(0) \neq 0 \quad \left(m' = \pm\frac{1}{2}\right).$$

Substituting (106) into (104) gives

$$a_{1/2,m'}(\vec{p}) = 0, \tag{107}$$

and (102) is thus simplified to (omitting the index 3/2 in $a_{3/2,m}(\vec{p})$)

$$a_{\lambda,r}(\vec{p}) = \left\langle 1,\lambda;\frac{1}{2},r \middle| 1,\frac{1}{2},\frac{3}{2},m \right\rangle a_m(\vec{p}). \tag{108a}$$

Similarly, (99b) leads to

$$b^+_{\lambda,r}(\vec{p}) = \left\langle 1,\lambda;\frac{1}{2},r \middle| 1,\frac{1}{2},\frac{3}{2},m \right\rangle b^+_m(\vec{p}). \tag{108b}$$

Inserting (108a) and (108b) in (98), we obtain

$$\Psi^\nu(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \left[a_m(\vec{p}) U^\nu_m(\vec{p}) \mathrm{e}^{\mathrm{i}px} + b^+_m(\vec{p}) V^\nu_m(\vec{p}) \mathrm{e}^{-\mathrm{i}px} \right], \tag{109}$$

where

$$U^\nu_m(\vec{p}) \equiv \sum_{\lambda=-1}^{1} \sum_{r=-1/2}^{1/2} e^\nu_\lambda(\vec{p}) u_r(\vec{p}) \delta(\lambda + r, m) \times \sqrt{\frac{\left(\frac{3}{2}+m\right)!\left(\frac{3}{2}-m\right)!}{3(1+\lambda)!(1-\lambda)!\left(\frac{1}{2}+r\right)!\left(\frac{1}{2}-r\right)!}}, \tag{110a}$$

$$V^\nu_m(\vec{p}) \equiv \sum_{\lambda=-1}^{1} \sum_{r=-1/2}^{1/2} \bar{e}^\nu_\lambda(\vec{p}) v_r(\vec{p}) \delta(\lambda + r, m) \times \sqrt{\frac{\left(\frac{3}{2}+m\right)!\left(\frac{3}{2}-m\right)!}{3(1+\lambda)!(1-\lambda)!\left(\frac{1}{2}+r\right)!\left(\frac{1}{2}-r\right)!}}. \tag{110b}$$

4.2 Spin 5/2

The R–S equations for spin 5/2, (38a)–(38f), might be rewritten as

$$(\Box - m^2)\Psi^{\nu_1\nu_2}(x) = 0, \quad \partial_{\nu_1}\Psi^{\nu_1\nu_2}(x) = 0,$$

$$\partial_{\nu_2}\Psi^{\nu_1\nu_2}(x)=0, \quad (111a)$$
$$\Psi^{\nu\nu}(x)=0,$$
$$\Psi^{\nu_1\nu_2}(x)=\Psi^{\nu_2\nu_1}(x), \quad (111b)$$
$$(\not\partial+m)\Psi^{\nu_1\nu_2}(x)=0, \quad (111c)$$
$$\gamma_\nu\Psi^{\nu\nu_2}(x)=0. \quad (111d)$$

Equations (111a) and (111b) could be solved in the same way in which (17a)–(17d) are solved; therefore, we have

$$\Psi^{\nu_1\nu_2}(x) \quad (112)$$
$$=\frac{1}{\sqrt{V}}\sum_{\vec{p}}[a_{\lambda_{12}}(\vec{p})e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p})\mathrm{e}^{\mathrm{i}px}+b^+_{\lambda_{12}}(\vec{p})\bar{e}^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p})\mathrm{e}^{-\mathrm{i}px}],$$

where

$$e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p})=\sum_{\lambda_1\lambda_2}e^{\nu_1}_{\lambda_1}(\vec{p})e^{\nu_2}_{\lambda_2}(\vec{p})\langle 1,\lambda_1;1,\lambda_2|1,1,2,\lambda_{12}\rangle, \quad (113a)$$

$$\bar{e}^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p})=g_{\nu_1\mu_1}g_{\nu_1\mu_2}\left(e^{\mu_1\mu_2}_{\lambda_{12}}(\vec{p})\right)^*$$
$$(\lambda_{12}=0,\pm1,\pm2). \quad (113b)$$

Substituting (112) into (111c) gives

$$(\mathrm{i}\not p+m)a_{\lambda_{12}}(\vec{p})=0,\quad (-\mathrm{i}\not p+m)b^+_{\lambda_{12}}(\vec{p})=0. \quad (114)$$

These are again the Dirac equations for spin 1/2, and their solutions are

$$a_{\lambda_{12}}(\vec{p})=u_r(\vec{p})a_{\lambda_{12},r}(\vec{p}),$$
$$\left(r=\frac{1}{2},-\frac{1}{2}\right), \quad (115a)$$
$$b^+_{\lambda_{12}}(\vec{p})=v_r(\vec{p})b^+_{\lambda_{12},r}(\vec{p}),$$
$$v_r(\vec{p})=\gamma_2u_r(\vec{p})^*. \quad (115b)$$

Inserting (115a) and (115b) into (112) yields

$$\Psi^{\nu_1\nu_2}(x)=\frac{1}{\sqrt{V}}\sum_{\vec{p}}\Big[e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p})u_r(\vec{p})a_{\lambda_{12},r}(\vec{p})\mathrm{e}^{\mathrm{i}px}$$
$$+\bar{e}^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p})v_r(\vec{p})b^+_{\lambda_{12},r}(\vec{p})\mathrm{e}^{-\mathrm{i}px}\Big]. \quad (116)$$

With this expression, (111d) becomes

$$\gamma_\nu e^{\nu\nu_2}_{\lambda_{12}}(\vec{p})u_r(\vec{p})a_{\lambda_{12},r}(\vec{p})=0, \quad (117a)$$
$$\gamma_\nu\bar{e}^{\nu\nu_2}_{\lambda_{12}}(\vec{p})v_r(\vec{p})b^+_{\lambda_{12},r}(\vec{p})=0. \quad (117b)$$

Utilizing (51), (95) and (100), (117a) can be rewritten as

$$\gamma_\nu e^{\nu\nu_2}_{\lambda_{12}}(0)u_r(0)a_{\lambda_{12},r}(\vec{p})=0, \quad (118)$$

where $a_{\lambda_{12},r}(\vec{p})$ is related to the two magnetic quantum numbers λ_{12} ($\lambda_{12}=2,1,0,-1,-2$) and r ($r=1/2,-1/2$), recalling the Clebsch–Gordan coefficients for coupling a spin-2 angular momentum to a spin-1/2 one, a general candidate for $a_{\lambda_{12},r}(\vec{p})$ is

$$a_{\lambda_{12},r}(\vec{p})=\sum_m\langle 2,\lambda_{12};\frac{1}{2},r|2,\frac{1}{2},\frac{5}{2},m\rangle a_{5/2,m}(\vec{p})$$
$$+\sum_{m'}\langle 2,\lambda_{12};1/2,r|2,\frac{1}{2},\frac{3}{2},m'\rangle a_{3/2,m'}(\vec{p}),$$
$$\left(m=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2};\quad m'=\pm\frac{1}{2},\pm\frac{3}{2}\right). \quad (119)$$

Let

$$U^{\nu_1\nu_2}_{5/2,m}(\vec{p})=\sum_{\lambda_{12}r}e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p})u_r(\vec{p})\langle 2,\lambda_{12};1/2,r|2,\frac{1}{2},\frac{5}{2},m\rangle, \quad (120a)$$

$$U^{\nu_1\nu_2}_{3/2,m'}(\vec{p})=\sum_{\lambda_{12}r}e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p})u_r(\vec{p})\langle 2,\lambda_{12};1/2,r|2,\frac{1}{2},\frac{3}{2},m'\rangle, \quad (120b)$$

then (118) takes the form

$$\gamma_\nu U^{\nu\nu_2}_{5/2,m}(0)a_{5/2,m}(\vec{p})+\gamma_\nu U^{\nu\nu_2}_{3/2,m'}(0)a_{3/2,m'}(\vec{p})=0. \quad (121)$$

With the aid of (105a)–(105c), a straightforward calculation gives

$$\gamma_\nu U^{\nu\nu_2}_{5/2,m}(0)=0\quad\left(m=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2}\right), \quad (122a)$$
$$\gamma_\nu U^{\nu\nu_2}_{3/2,m'}(0)$$
$$=\mathrm{i}\sqrt{\frac{5}{2}}\gamma_5\sum_{\lambda_2r}e^{\nu_2}_{\lambda_2}(0)u_r(0)\langle 1,\lambda_2;\frac{1}{2},r|1,\frac{1}{2},\frac{3}{2},m'\rangle\neq 0,$$
$$\left(m'=\pm\frac{1}{2},\pm\frac{3}{2}\right). \quad (122b)$$

Substituting (122a) and (122b) into (121) gives

$$a_{3/2,m'}(\vec{p})=0. \quad (123)$$

Thus (119) is simplified to (omitting the index 5/2 in $a_{5/2,m}(\vec{p})$)

$$a_{\lambda_{12},r}(\vec{p})=\sum_m\left\langle 2,\lambda_{12};1/2,r\middle|2,\frac{1}{2},\frac{5}{2},m\right\rangle a_m(\vec{p}). \quad (124a)$$

Similarly, (117b) leads to

$$b^+_{\lambda_{12},r}(\vec{p})=\sum_m\left\langle 2,\lambda_{12};\frac{1}{2},r\middle|2,\frac{1}{2},\frac{5}{2},m\right\rangle b^+_m(\vec{p}). \quad (124b)$$

Substituting (124a) and (124b) back into (116), we finally obtain

$$\Psi^{\nu_1\nu_2}(x)=\frac{1}{\sqrt{V}}\sum_{\vec{p}}\Big[a_m(\vec{p})U^{\nu_1\nu_2}_m(\vec{p})\mathrm{e}^{\mathrm{i}px}$$
$$+b_m+(\vec{p})V^{\nu_1\nu_2}_m(\vec{p})\mathrm{e}^{-\mathrm{i}px}\Big], \quad (125)$$

with

$$U^{\nu_1\nu_2}_m(\vec{p})$$

$$= \sum_{\lambda_{12} r} e^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p}) u_r(\vec{p}) \left\langle 2, \lambda_{12}; \frac{1}{2}, r \middle| 2, \frac{1}{2}, \frac{5}{2}, m \right\rangle \tag{126}$$

$$\equiv \sum_{\lambda_1,\lambda_2=-1}^{1} \sum_{r=-1/2}^{1/2} e^{\nu_1}_{\lambda_1}(\vec{p}) e^{\nu_2}_{\lambda_2}(\vec{p}) u_r(\vec{p}) \delta(\lambda_1+\lambda_2+r, m)$$

$$\times \sqrt{\frac{\left(\frac{5}{2}+m\right)!\left(\frac{5}{2}-m\right)!}{30(1+\lambda_1)!(1-\lambda_1)!(1+\lambda_2)!(1-\lambda_2)!\left(\frac{1}{2}+r\right)!\left(\frac{1}{2}-r\right)!}},$$

and

$$V^{\nu_1\nu_2}_m(\vec{p})$$

$$= \sum_{\lambda_{12} r} \bar{e}^{\nu_1\nu_2}_{\lambda_{12}}(\vec{p}) v_r(\vec{p}) \left\langle 2, \lambda_{12}; \frac{1}{2}, r \middle| 2, \frac{1}{2}, \frac{5}{2}, m \right\rangle \tag{127}$$

$$\equiv \sum_{\lambda_1,\lambda_2=-1}^{1} \sum_{r=-\frac{1}{2}}^{1/2} \bar{e}^{\nu_1}_{\lambda_1}(\vec{p}) \bar{e}^{\nu_2}_{\lambda_2}(\vec{p}) v_r(\vec{p}) \delta(\lambda_1+\lambda_2+r, m)$$

$$\times \sqrt{\frac{\left(\frac{5}{2}+m\right)!\left(\frac{5}{2}-m\right)!}{30(1+\lambda_1)!(1-\lambda_1)!(1+\lambda_2)!(1-\lambda_2)!\left(\frac{1}{2}+r\right)!\left(\frac{1}{2}-r\right)!}}.$$

4.3 Spin $n + 1/2$

Finally, the R–S equations for half-integral spin-$n + 1/2$ (41a)–(41f) are to be solved. The first four equations, (41a)–(41d), which are similar to that for spin n [refer to (28a)–(28d)], lead to an intermediate result:

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \left[e^{\nu_1\nu_2\ldots\nu_n}_M(\vec{p}) a_M(\vec{p}) \mathrm{e}^{\mathrm{i}px} + \bar{e}^{\nu_1\nu_2\ldots\nu_n}_M(\vec{p}) b^+_M(\vec{p}) \mathrm{e}^{-\mathrm{i}px} \right], \tag{128}$$

where $M = 0, \pm1, \pm2, \cdots, \pm n$, $e^{\nu_1\nu_2\ldots\nu_n}_M(\vec{p})$ and $\bar{e}^{\nu_1\nu_2\ldots\nu_n}_M(\vec{p})$ are the wave functions for an integral spin n and take the forms as expressed in (90a) and (90b) or (91) except that the momentum $\vec{k}$ is now being replaced by $\vec{p}$. Substituting (128) into (41e) yields

$$(\mathrm{i}\,\not{p} + m) a_M(\vec{p}) = 0, \quad (-\mathrm{i}\,\not{p} + m) b^+_M(\vec{p}) = 0. \tag{129}$$

These are Dirac equations for spin 1/2, and their solutions are

$$a_M(\vec{p}) = u_r(\vec{p}) a_{M,r}(\vec{p}), \quad b^+_M(\vec{p}) = v_r(\vec{p}) b^+_{M,r}(\vec{p}) \quad \left(r = \frac{1}{2}, -\frac{1}{2}\right). \tag{130}$$

Inserting (130) in (128) gives

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \left[e^{\nu_1\nu_2\ldots\nu_n}_M(\vec{p}) u_r(\vec{p}) a_{M,r}(\vec{p}) \mathrm{e}^{\mathrm{i}px} + \bar{e}^{\nu_1\nu_2\ldots\nu_n}_M(\vec{p}) v_r(\vec{p}) b^+_{M,r}(\vec{p}) \mathrm{e}^{-\mathrm{i}px} \right]. \tag{131}$$

With this expression, (41f) becomes

$$\gamma_\nu e^{\nu\nu_2\cdots\nu_n}_M(\vec{p}) u_r(\vec{p}) a_{M,r}(\vec{p}) = 0, \tag{132a}$$

$$\gamma_\nu \bar{e}^{\nu\nu_2\cdots\nu_n}_M(\vec{p}) v_r(\vec{p}) b^+_{M,r}(\vec{p}) = 0. \tag{132b}$$

Equation (132a) can be rewritten as

$$\gamma_\nu e^{\nu\nu_2\cdots\nu_n}_M(0) u_r(0) a_{M,r}(\vec{p}) = 0, \tag{133}$$

where $a_{M,r}(\vec{p})$ is related to the two magnetic quantum numbers M ($M = 0, \pm1, \pm2, \cdots, \pm n$) and r ($r = 1/2, -1/2$); recalling the Clebsch–Gordan coefficients for coupling a spin-n angular momentum to a spin-1/2 one, a general candidate for $a_{M,r}(\vec{p})$ is

$$a_{M,r}(\vec{p}) = \sum_m \left\langle n, M; \frac{1}{2}, r \middle| n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle a_{n+1/2,m}(\vec{p}) + \sum_{m'} \left\langle n, M; \frac{1}{2}, r \middle| n, \frac{1}{2}, n-\frac{1}{2}, m' \right\rangle a_{n-1/2,m'}(\vec{p})$$

$$\left(m = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\left(n+\frac{1}{2}\right); \quad m' = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\left(n-\frac{1}{2}\right) \right). \tag{134}$$

Let

$$U^{\nu_1\nu_2\cdots\nu_n}_{n+1/2,m}(\vec{p}) = \sum_{M,r} e^{\nu_1\nu_2\cdots\nu_n}_M(\vec{p}) u_r(\vec{p}) \left\langle n, M; \frac{1}{2}, r \middle| n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle, \tag{135a}$$

$$U^{\nu_1\nu_2\cdots\nu_n}_{n-1/2,m'}(\vec{p}) = \sum_{M,r} e^{\nu_1\nu_2\cdots\nu_n}_M(\vec{p}) u_r(\vec{p}) \left\langle n, M; \frac{1}{2}, r \middle| n, \frac{1}{2}, n-\frac{1}{2}, m' \right\rangle, \tag{135b}$$

then (133) takes the form

$$\gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n+1/2,m}(0) a_{n+1/2,m}(\vec{p}) + \gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n-1/2,m}(0) a_{n-1/2,m}(\vec{p}) = 0. \tag{136}$$

However,

$$\gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n+1/2,m}(0) = 0 \left(m = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\left(n+\frac{1}{2}\right) \right), \tag{137a}$$

$$\gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n-1/2,m'}(0) = \mathrm{i}\gamma_5 \sqrt{\frac{2n+1}{n}} U^{\nu_2\cdots\nu_n}_{n-1/2,m'}(0) \neq 0 \quad \left(m' = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\left(n-\frac{1}{2}\right) \right). \tag{137b}$$

Substituting (137a) and (137b) into (136) yields

$$a_{n-1/2,m'}(\vec{p}) = 0, \tag{138}$$

and (134) becomes (omitting the index $n+1/2$ in $a_{n+1/2,m}(\vec{p})$)

$$a_{M,r}(\vec{p}) = \sum_m \left\langle n, M; \frac{1}{2}, r \middle| n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle a_m(\vec{p}). \tag{139a}$$

Similarly, (132b) leads to

$$b^+_{M,r}(\vec{p}) = \sum_m \left\langle n, M; \frac{1}{2}, r \middle| n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle b^+_m(\vec{p}). \quad (139b)$$

Inserting (139a) and (139b) in (131) gives the final result

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \Big[a_m(\vec{p}) U_m^{\nu_1\nu_2\cdots\nu_n}(\vec{p}) \mathrm{e}^{\mathrm{i}px} + b^+_m(\vec{p}) V_m^{\nu_1\nu_2\cdots\nu_n}(\vec{p}) \mathrm{e}^{-\mathrm{i}px} \Big], \quad (140)$$

with

$$\begin{aligned} U_m^{\nu_1\nu_2\cdots\nu_n}(\vec{p}) &= \sum_{M,r} e_M^{\nu_1\nu_2\cdots\nu_n}(\vec{p}) u_r(\vec{p}) \left\langle n, M; \frac{1}{2}, r \middle| n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle \\ &= \sqrt{\frac{n+\frac{1}{2}+m}{2n+1}} e^{\nu_1\nu_2\cdots\nu_n}_{m-1/2}(\vec{p}) u_{1/2}(\vec{p}) \\ &+ \sqrt{\frac{n+\frac{1}{2}-m}{2n+1}} e^{\nu_1\nu_2\cdots\nu_n}_{m+1/2}(\vec{p}) u_{-1/2}(\vec{p}), \quad (141a) \end{aligned}$$

$$\begin{aligned} V_m^{\nu_1\nu_2\cdots\nu_n}(\vec{p}) &= \sum_{M,r} \bar{e}_M^{\nu_1\nu_2\cdots\nu_n}(\vec{p}) v_r(\vec{p}) \left\langle n, M; \frac{1}{2}, r \middle| n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle \\ &= \sqrt{\frac{n+\frac{1}{2}+m}{2n+1}} \bar{e}^{\nu_1\nu_2\cdots\nu_n}_{m-1/2}(\vec{p}) v_{1/2}(\vec{p}) \\ &+ \sqrt{\frac{n+\frac{1}{2}-m}{2n+1}} \bar{e}^{\nu_1\nu_2\cdots\nu_n}_{m+1/2}(\vec{p}) v_{-1/2}(\vec{p}). \quad (141b) \end{aligned}$$

In order to illustrate these formulas, we give explicit expressions for the wave functions of spins 5/2 an 7/2. For spin 5/2, the positive energy wave functions are

$$\begin{aligned} U^{\nu_1\nu_2}_{5/2} &= e^{\nu_1}_{+1} e^{\nu_2}_{+1} u_{1/2}, \\ U^{\nu_1\nu_2}_{3/2} &= \sqrt{\frac{1}{5}} e^{\nu_1}_{+1} e^{\nu_2}_{+1} u_{-1/2} + \sqrt{\frac{2}{5}} e^{\nu_1}_{+1} e^{\nu_2}_{0} u_{1/2} \\ &+ \sqrt{\frac{2}{5}} e^{\nu_1}_{0} e^{\nu_2}_{+1} u_{1/2}, \\ U^{\nu_1\nu_2}_{1/2} &= \sqrt{\frac{1}{5}} e^{\nu_1}_{+1} e^{\nu_2}_{0} u_{-1/2} + \sqrt{\frac{1}{5}} e^{\nu_1}_{0} e^{\nu_2}_{+1} u_{-1/2} \\ &+ \sqrt{\frac{1}{10}} e^{\nu_1}_{+1} e^{\nu_2}_{-1} u_{1/2} \\ &+ \sqrt{\frac{4}{10}} e^{\nu_1}_{0} e^{\nu_2}_{0} u_{1/2} + \sqrt{\frac{1}{10}} e^{\nu_1}_{-1} e^{\nu_2}_{+1} u_{1/2}, \\ U^{\nu_1\nu_2}_{-1/2} &= \sqrt{\frac{1}{10}} e^{\nu_1}_{+1} e^{\nu_2}_{-1} u_{-1/2} + \sqrt{\frac{4}{10}} e^{\nu_1}_{0} e^{\nu_2}_{0} u_{-1/2} \\ &+ \sqrt{\frac{1}{10}} e^{\nu_1}_{-1} e^{\nu_2}_{+1} u_{-1/2} + \sqrt{\frac{1}{5}} e^{\nu_1}_{0} e^{\nu_2}_{-1} u_{1/2} \\ &+ \sqrt{\frac{1}{5}} e^{\nu_1}_{-1} e^{\nu_2}_{0} u_{1/2}, \\ U^{\nu_1\nu_2}_{-3/2} &= \sqrt{\frac{2}{5}} e^{\nu_1}_{0} e^{\nu_2}_{-1} u_{-1/2} + \sqrt{\frac{2}{5}} e^{\nu_1}_{-1} e^{\nu_2}_{0} u_{-1/2} \\ &+ \sqrt{\frac{1}{5}} e^{\nu_1}_{-1} e^{\nu_2}_{-1} u_{1/2}, \\ U^{\nu_1\nu_2}_{-5/2} &= e^{\nu_1}_{-1} e^{\nu_2}_{-1} u_{-1/2}, \end{aligned}$$

and for spin 7/2, the positive energy wave functions take on the form

$$\begin{aligned} U^{\nu_1\nu_2\nu_3}_{7/2} &= e^{\nu_1}_{+1} e^{\nu_2}_{+1} e^{\nu_3}_{+1} u_{1/2}, \\ U^{\nu_1\nu_2\nu_3}_{5/2} &= \sqrt{\frac{2}{7}} [e^{\nu_1}_{+1} e^{\nu_2}_{+1} e^{\nu_3}_{0} + e^{\nu_1}_{+1} e^{\nu_2}_{0} e^{\nu_3}_{+1} + e^{\nu_1}_{0} e^{\nu_2}_{+1} e^{\nu_3}_{+1}] u_{1/2} \\ &+ \sqrt{\frac{1}{7}} e^{\nu_1}_{+1} e^{\nu_2}_{+1} e^{\nu_3}_{+1} u_{-1/2}, \\ U^{\nu_1\nu_2\nu_3}_{3/2} &= \sqrt{\frac{1}{21}} \Big[e^{\nu_1}_{+1} e^{\nu_2}_{+1} e^{\nu_3}_{-1} + e^{\nu_1}_{+1} e^{\nu_2}_{-1} e^{\nu_3}_{+1} + e^{\nu_1}_{-1} e^{\nu_2}_{+1} e^{\nu_3}_{+1} \\ &+ 2e^{\nu_1}_{+1} e^{\nu_2}_{0} e^{\nu_3}_{0} + 2e^{\nu_1}_{0} e^{\nu_2}_{+1} e^{\nu_3}_{0} + 2e^{\nu_1}_{0} e^{\nu_2}_{0} e^{\nu_3}_{+1} \Big] u_{1/2} \\ &+ \sqrt{\frac{2}{21}} \Big[e^{\nu_1}_{+1} e^{\nu_2}_{+1} e^{\nu_3}_{0} + e^{\nu_1}_{+1} e^{\nu_2}_{0} e^{\nu_3}_{+1} \\ &+ e^{\nu_1}_{0} e^{\nu_2}_{+1} e^{\nu_3}_{+1} \Big] u_{-1/2}, \\ U^{\nu_1\nu_2\nu_3}_{1/2} &= \sqrt{\frac{2}{35}} \Big[e^{\nu_1}_{+1} e^{\nu_2}_{0} e^{\nu_3}_{-1} + e^{\nu_1}_{0} e^{\nu_2}_{+1} e^{\nu_3}_{-1} + e^{\nu_1}_{+1} e^{\nu_2}_{-1} e^{\nu_3}_{0} \\ &+ 2e^{\nu_1}_{0} e^{\nu_2}_{0} e^{\nu_3}_{0} + e^{\nu_1}_{-1} e^{\nu_2}_{+1} e^{\nu_3}_{0} + e^{\nu_1}_{0} e^{\nu_2}_{-1} e^{\nu_3}_{+1} \\ &+ e^{\nu_1}_{-1} e^{\nu_2}_{0} e^{\nu_3}_{+1} \Big] u_{1/2} + \sqrt{\frac{1}{35}} \Big[e^{\nu_1}_{+1} e^{\nu_2}_{+1} e^{\nu_3}_{-1} \\ &+ e^{\nu_1}_{+1} e^{\nu_2}_{-1} e^{\nu_3}_{+1} + e^{\nu_1}_{-1} e^{\nu_2}_{+1} e^{\nu_3}_{+1} + 2e^{\nu_1}_{+1} e^{\nu_2}_{0} e^{\nu_3}_{0} \\ &+ 2e^{\nu_1}_{0} e^{\nu_2}_{+1} e^{\nu_3}_{0} + 2e^{\nu_1}_{0} e^{\nu_2}_{0} e^{\nu_3}_{+1} \Big] u_{-1/2}, \\ U^{\nu_1\nu_2\nu_3}_{-1/2} &= \sqrt{\frac{1}{35}} \Big[2e^{\nu_1}_{0} e^{\nu_2}_{0} e^{\nu_3}_{-1} + 2e^{\nu_1}_{0} e^{\nu_2}_{-1} e^{\nu_3}_{0} + 2e^{\nu_1}_{-1} e^{\nu_2}_{0} e^{\nu_3}_{0} \\ &+ e^{\nu_1}_{+1} e^{\nu_2}_{-1} e^{\nu_3}_{-1} + e^{\nu_1}_{-1} e^{\nu_2}_{+1} e^{\nu_3}_{-1} + e^{\nu_1}_{-1} e^{\nu_2}_{-1} e^{\nu_3}_{+1} \Big] u_{1/2} \\ &+ \sqrt{\frac{2}{35}} \Big[e^{\nu_1}_{+1} e^{\nu_2}_{0} e^{\nu_3}_{-1} + e^{\nu_1}_{0} e^{\nu_2}_{+1} e^{\nu_3}_{-1} + e^{\nu_1}_{+1} e^{\nu_2}_{-1} e^{\nu_3}_{0} \\ &+ 2e^{\nu_1}_{0} e^{\nu_2}_{0} e^{\nu_3}_{0} + e^{\nu_1}_{-1} e^{\nu_2}_{+1} e^{\nu_3}_{0} + e^{\nu_1}_{0} e^{\nu_2}_{-1} e^{\nu_3}_{+1} \\ &+ e^{\nu_1}_{-1} e^{\nu_2}_{0} e^{\nu_3}_{+1} \Big] u_{-1/2}, \\ U^{\nu_1\nu_2\nu_3}_{-3/2} &= \sqrt{\frac{2}{21}} [e^{\nu_1}_{0} e^{\nu_2}_{-1} e^{\nu_3}_{-1} + e^{\nu_1}_{-1} e^{\nu_2}_{0} e^{\nu_3}_{-1} + e^{\nu_1}_{-1} e^{\nu_2}_{-1} e^{\nu_3}_{0}] u_{1/2} \\ &+ \sqrt{\frac{1}{21}} \Big[2e^{\nu_1}_{0} e^{\nu_2}_{0} e^{\nu_3}_{-1} + 2e^{\nu_1}_{0} e^{\nu_2}_{-1} e^{\nu_3}_{0} \\ &+ 2e^{\nu_1}_{-1} e^{\nu_2}_{0} e^{\nu_3}_{0} + e^{\nu_1}_{+1} e^{\nu_2}_{-1} e^{\nu_3}_{-1} \\ &+ e^{\nu_1}_{-1} e^{\nu_2}_{+1} e^{\nu_3}_{-1} + e^{\nu_1}_{-1} e^{\nu_2}_{-1} e^{\nu_3}_{+1} \Big] u_{-1/2}, \\ U^{\nu_1\nu_2\nu_3}_{-5/2} &= \sqrt{\frac{1}{7}} e^{\nu_1}_{-1} e^{\nu_2}_{-1} e^{\nu_3}_{-1} u_{1/2} + \sqrt{\frac{2}{7}} \Big[e^{\nu_1}_{0} e^{\nu_2}_{-1} e^{\nu_3}_{-1} \\ &+ e^{\nu_1}_{-1} e^{\nu_2}_{0} e^{\nu_3}_{-1} + e^{\nu_1}_{-1} e^{\nu_2}_{-1} e^{\nu_3}_{0} \Big] u_{-1/2}, \end{aligned}$$

$$U_{-7/2}^{\nu_1\nu_2\nu_3} = e_{-1}^{\nu_1} e_{-1}^{\nu_2} e_{-1}^{\nu_3} u_{-1/2}.$$

The negative energy wave functions, $V_m^{\nu_1\nu_2}$ and $V_m^{\nu_1\nu_2\nu_3}$, can be listed by replacing e_λ^ν with $\bar{e}_\lambda^\nu$ and u_r with v_r in the positive energy wave functions.

The main points for solving the R–S equations for half-integral spin are as follows. With spin increasing from n to $n+1/2$ by half a unit, two new equations are added to the R–S equations for spin $n+1/2$; one is the Dirac equation $(\not\partial + m)\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0$, the other is the subsidiary condition $\gamma_\nu \Psi^{\nu\nu_2\nu_3\cdots\nu_n}(x) = 0$. The Dirac equation leads to a coupling between spin-n wave functions and spin-1/2 wave functions, which results in two classes of different total spin wave functions, namely wave functions for spin $n+1/2$ and $n-1/2$. The subsidiary condition removes the spin $n-1/2$ wave functions. Thus, only the spin $n+1/2$ wave functions, corresponding to the maximum possible spin, are kept in the final expression of the solution.

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Appendix

A The explicit matrix form of the Lorentz transformation

We can write $L = \mathrm{e}^{-\mathrm{i}S_3\phi}\mathrm{e}^{-\mathrm{i}S_2\theta}\mathrm{e}^{\varepsilon K_3}$;

$$L = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & (\varpi/m)\sin\theta\cos\phi & -\mathrm{i}k_1/m \\ \cos\theta\sin\phi & \cos\phi & (\varpi/m)\sin\theta\sin\phi & -\mathrm{i}k_2/m \\ -\sin\theta & 0 & (\varpi/m)\cos\theta & -\mathrm{i}k_3/m \\ 0 & 0 & \mathrm{i}|\vec{k}|/m & \varpi/m \end{pmatrix},$$

for particles with half-integral spin, ϖ and $\vec{k}$ are replaced by E and $\vec{p}$, respectively.

B The explicit expression of (56a)–(56c)

We have

$$\begin{aligned}
e_{22}^{\nu_1\nu_2}(0) &= e_{+1}^{\nu_1}(0)e_{+1}^{\nu_2}(0), \\
e_{21}^{\nu_1\nu_2}(0) &= \frac{1}{\sqrt{2}}[e_{+1}^{\nu_1}(0)e_0^{\nu_2}(0) + e_0^{\nu_1}(0)e_{+1}^{\nu_2}(0)], \\
e_{20}^{\nu_1\nu_2}(0) &= \frac{1}{\sqrt{6}}\Big[e_{+1}^{\nu_1}(0)e_{-1}^{\nu_2}(0) \\
&\quad + 2e_0^{\nu_1}(0)e_0^{\nu_2}(0) + e_{-1}^{\nu_1}(0)e_{+1}^{\nu_2}(0)\Big], \\
e_{2-1}^{\nu_1\nu_2}(0) &= \frac{1}{\sqrt{2}}[e_0^{\nu_1}(0)e_{-1}^{\nu_2}(0) + e_{-1}^{\nu_1}(0)e_0^{\nu_2}(0)], \\
e_{2-2}^{\nu_1\nu_2}(0) &= e_{-1}^{\nu_1}(0)e_{-1}^{\nu_2}(0), \\
e_{11}^{\nu_1\nu_2}(0) &= \frac{1}{\sqrt{2}}[e_{+1}^{\nu_1}(0)e_0^{\nu_2}(0) - e_0^{\nu_1}(0)e_{+1}^{\nu_2}(0)], \\
e_{10}^{\nu_1\nu_2}(0) &= \frac{1}{\sqrt{2}}[e_{+1}^{\nu_1}(0)e_{-1}^{\nu_2}(0) - e_{-1}^{\nu_1}(0)e_{+1}^{\nu_2}(0)], \\
e_{1-1}^{\nu_1\nu_2}(0) &= \frac{1}{\sqrt{2}}[e_0^{\nu_1}(0)e_{-1}^{\nu_2}(0) - e_{-1}^{\nu_1}(0)e_0^{\nu_2}(0)], \\
e_{00}^{\nu_1\nu_2}(0) &= \frac{1}{\sqrt{3}}\Big[e_{+1}^{\nu_1}(0)e_{-1}^{\nu_2}(0) - e_0^{\nu_1}(0)e_0^{\nu_2}(0) \\
&\quad + e_{-1}^{\nu_1}(0)e_{+1}^{\nu_2}(0)\Big].
\end{aligned}$$

C The explicit expression of (77a)–(77c)

Denoting $e_{\lambda_1}^{\nu_1}(0)e_{\lambda_2}^{\nu_2}(0)e_{\lambda_3}^{\nu_3}(0) \equiv (\lambda_1,\lambda_2,\lambda_3)$, $(\lambda_1,\lambda_2,\lambda_3 = 1,0,-1)$, the explicit form of (37) can be expressed by

$$\begin{aligned}
e_{33}^{\nu_1\nu_2\nu_3}(0) &= (1,1,1), \\
e_{32}^{\nu_1\nu_2\nu_3}(0) &= \frac{1}{\sqrt{3}}[(1,1,0)+(1,0,1)+(0,1,1)], \\
e_{31}^{\nu_1\nu_2\nu_3}(0) &= \frac{1}{\sqrt{15}}\Big[(1,1,-1)+2(1,0,0)+2(0,1,0) \\
&\quad + (1,-1,1)+2(0,0,1)+(-1,1,1)\Big], \\
e_{30}^{\nu_1\nu_2\nu_3}(0) &= \frac{1}{\sqrt{10}}\Big[(1,0,-1)+(0,1,-1)+(1,-1,0) \\
&\quad + 2(0,0,0)+(-1,1,0)+(0,-1,1) \\
&\quad + (-1,0,1)\Big], \\
e_{3,-1}^{\nu_1\nu_2\nu_3}(0) &= \frac{1}{\sqrt{15}}\Big[(1,-1,-1)+2(0,0,-1) \\
&\quad + (-1,1,-1) \\
&\quad + 2(0,-1,0)+2(-1,0,0)+(-1,-1,1)\Big], \\
e_{3,-2}^{\nu_1\nu_2\nu_3}(0) &= \frac{1}{\sqrt{3}}\Big[(0,-1,-1)+(-1,0,-1) \\
&\quad + (-1,-1,0)\Big], \\
e_{3,-3}^{\nu_1\nu_2\nu_3}(0) &= (-1,-1,-1), \\
e_{2,2}^{\nu_1\nu_2\nu_3}(0) &= \sqrt{\frac{2}{3}}(1,1,0) - \frac{1}{\sqrt{6}}[(1,0,1)+(0,1,1)], \\
e_{2,1}^{\nu_1\nu_2\nu_3}(0) &= \frac{1}{\sqrt{3}}(1,1,-1) + \frac{1}{\sqrt{12}}\Big[(1,0,0)+(0,1,0) \\
&\quad - (1,-1,1) - 2(0,0,1) - (-1,1,1)\Big], \\
e_{2,0}^{\nu_1\nu_2\nu_3}(0) &= \frac{1}{2}\Big[(1,0,-1)+(0,1,-1)-(0,-1,1) \\
&\quad - (-1,0,1)\Big], \\
e_{2,-1}^{\nu_1\nu_2\nu_3}(0) &= \frac{1}{\sqrt{12}}\Big[(1,-1,-1)+2(0,0,-1) \\
&\quad + (-1,1,-1)-(0,-1,0)-(-1,0,0)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{\sqrt{3}}(-1,-1,1), \\
e^{\nu_1\nu_2\nu_3}_{2,-2}(0) &= \frac{1}{\sqrt{6}}[(0,-1,-1)+(-1,0,-1)] \\
&\quad - \sqrt{\frac{2}{3}}(-1,-1,0), \\
e^{\nu_1\nu_2\nu_3}_{1,1}(0) &= \sqrt{\frac{3}{5}}(1,1,-1) - \sqrt{\frac{3}{20}}[(1,0,0)+(0,1,0)] \\
&\quad + \sqrt{\frac{1}{60}}[(1,-1,1)+2(0,0,1)+(-1,1,1)], \\
e^{\nu_1\nu_2\nu_3}_{1,0}(0) &= \sqrt{\frac{3}{20}}\Big[(1,0,-1)+(0,1,-1)+(0,-1,1) \\
&\quad + (-1,0,1)\Big] - \sqrt{\frac{1}{15}}\Big[(1,-1,0)+2(0,0,0) \\
&\quad + (-1,1,0)\Big], \\
e^{\nu_1\nu_2\nu_3}_{1,-1}(0) &= \sqrt{\frac{3}{5}}(-1,-1,1) - \sqrt{\frac{3}{20}}\Big[(-1,0,0) \\
&\quad + (0,-1,0)\Big] + \sqrt{\frac{1}{60}}\Big[(1,-1,-1) \\
&\quad + 2(0,0,-1)+(-1,1,-1)\Big].
\end{aligned}
$$

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